Fondements Géométriques de la Mécanique des Milieux Continus

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"Matériaux et Symétries Matérielles"

Nantes, 08-10 Septembre 2014

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Part I

Galilean Mechanics

Chapter 1

Galileo's Principle of Relativity

1.1 Events and space-time

Definition 1.1 *An **event** $\mathbf{X}$ is just an occurence at a specific moment and at a specific place. The **space-time** (or **universe**) is the set $\mathcal{U}$ of all the events.*

The lightning stricking a tree, a crash, the battle of Fontenoy, a birthday, the reception of an e-mail by a computer are some examples of events. Most of the events are relatively blurred, without either beginning or end or precisely defined localization. The events which, within the limits imposed by our measuring instruments, seem instantaneous and pointwise are called **punctual events**. In the sequel, when talking about events, the reader is referred only to punctual events.

Definition 1.2 *A **particle** is an object appearing as a pointwise phenomenon endowed with some time persistence.*

We can see it as a sequence of events. A trace can be kept for instance thanks to a film consisting of frames recorded by a camera. Of course this kind of observations has a discontinuous feature. If a high-speed camera is used, the observed events are closer. If we imagine as the time resolution can be arbitrarely reduced, a continuous sequence of events is obtained.

Definition 1.3 *a **trajectory** is the continuous sequence of events revealing the persistency of a particle and represented by a continuous map $t \mapsto \mathbf{X}(t)$.*

1.2 Event coordinates

1.2.1 When?

The **clock** is an instrument allowing to measure the **durations**.

Definition 1.4 *By the choice of a reference event* $\mathbf{X}_0$ *to which the time* $t_0 = 0$ *is assigned, an observer can assign to any event* $\mathbf{X}$ *a number* t *called the* ***date****, equal to the duration between* $\mathbf{X}_0$ *and* $\mathbf{X}$*, if* $\mathbf{X}$ *succeeds to* $\mathbf{X}_0$*, and to its opposite, if* $\mathbf{X}$ *precedes* $\mathbf{X}_0$ *.*

Conversely, the duration elapsed between two events $\mathbf{X}_1$ and $\mathbf{X}_2$ is calculated as the date difference $\Delta t = t_2 - t_1$. We assume that all the clocks are synchronized, *i.e.* they measure the same duration between any events:

$$\Delta t = \Delta t' \ . \tag{1.1}$$

This means each clock measures the durations with the same unit (the second for instance). This entails also that if a clock assigns a date t' to some event, the other one assigns to the same event a date $t = t' + \tau$ where τ depends only on both clocks.

Definition 1.5 *two events are* ***simultaneous*** *if, measured with the same clock, their dates are identical.*

Clearly, if two events are simultaneous for a clock, it is so for any other one.

1.2.2 Where?

The most common measuring instrument for a **distance** is the **graduated ruler**. Of course, there exist less accurate instruments (the land-surveyor's string or measuring tape) while other ones are much more (especially thanks to the lasers) but, for the simplicity of the presentation, the reader is only referred to the rulers as distance measuring instruments.

Whatever, we have just to know that the ruler allows us measuring the distance Δs between two *simultaneous events* $\mathbf{X}_1$ and $\mathbf{X}_2$. We assume all the rulers are standardized in the sense they measure the same distance between events:

$$\Delta s = \Delta s' \ . \tag{1.2}$$

This means each ruler measures the durations with the same unit (the metre for instance). Let us have a break now to explain the meaning of the simultaneity between events. When they fit the ruler graduations, the observer is informed by light signals. The essential point is –as said before– these signals arrive to the observer with an infinite velocity, then instantaneously.

As we assigned to each event a date, we would like to assign it a position. Whithout entering into details of the measurement method which are not useful to our discussion, let say only that –in addition to the rulers– instruments are required to measure the angles, for instance **set squares** and **protractors**. We admit the measurement method allows an observer assigning to any event $\mathbf{X}$ three coordinates r^1, r^2, r^3. The column vector gathering them:

$$r = \begin{pmatrix} r^1 \\ r^2 \\ r^3 \end{pmatrix} ,$$

is called **position** of **X**.

Definition 1.6 *to each event* **X***, an observer can assign a time* t *–in the sense defined by 1.4– and a column* $r \in \mathbb{R}^3$*, called the* ***position****, by the choice of a reference event* $\mathbf{X}_0$ *with position* $r_0 = 0$ *and in such way that for any distinct but simultaneous events* $\mathbf{X}_1$*,* $\mathbf{X}_2$ *and* $\mathbf{X}_3$ *of respective positions* r_1*,* r_2 *and* r_3*,*

- *if* $\Delta r = r_2 - r_1$*, we can calculate the distance between the first two by:*

$$\Delta s = \parallel \Delta r \parallel \ ,$$

- *and the angle* θ *between the segments* Δr *and* $\Delta' r = r_3 - r_1$ *by:*

$$\cos(\theta) = \Delta r \cdot \Delta' r \ / \ \parallel \Delta r \parallel \parallel \Delta' r \parallel \ .$$

In short, any observer has available instruments measuring durations, distances and angles. This allows him assigning to each event **X** a date t and a position r. In the sequel, we adopt the following convention:

Convention 1.1 *Coordinate labels:*

- *Latin indices* i, j, k *and so on run over the spacial coordinate labels, usually,* $1, 2, 3$ *or* x, y, z*.*
- *Greek indices* α, β, γ *and so on run over the four space-time coordinate labels* $0, 1, 2, 3$ *or* t, x, y, z*.*

Definition 1.7 *to each event* **X***, a column* $X \in \mathbb{R}^4$*:*

$$X = \begin{pmatrix} t \\ r \end{pmatrix} = \begin{pmatrix} t \\ r^1 \\ r^2 \\ r^3 \end{pmatrix} ,$$

is assigned by an observer. Their components $X^0 = t$*,* $X^i = r^i$ *are called* ***coordinates*** *of the event. The assignment is one-to-one. Each observer creates his own* ***coordinate system****.*

Hence, an observer can record the trajectory of a particle $t \mapsto \mathbf{X}(t)$ thanks to an assignment $t \mapsto X(t)$ in his own coordinate system.

Additionally, the length and angle measures allow to calculate the areas and volumes, at least for simple geometrical objects.

Definition 1.8 *The positions being determined by an observer for simultaneous events:*

- *the positions of three of its vertices being r_1, r_2, r_3, the **area** of a parallelogram is calculated by:*

$$S = \| \Delta r \times \Delta' r \| \ ,$$

with $\Delta r = r_2 - r_1$ and $\Delta' r = r_3 - r_1$;

- *three adjoining faces of it being defined by four of its vertices r_1, r_2, r_3, r_4, the **oriented volume** of a parallelepiped is calculated by:*

$$V = (\Delta r \times \Delta' r) \cdot \Delta'' r,$$

with$\Delta'' r = r_4 - r_1$.

1.3 Galilean transformations

1.3.1 Uniform straight motion

Newton's first law claims the velocity of a particle or a body remains constant unless the body is acted upon by an external force. This assumes we know what is a force, at least intuitively. We prefer to take it as starting point to define the forces.

Definition 1.9 *A **force** is a phenomenon modifying the velocity of a particle. Hence a force **free particle** moves in a straight line at uniform velocity. This is the **uniform straight motion (USM)**. If the velocity is null, the particle is said to be **at rest** in the considered coordinate system.*

The problem is that gravity is a large scale force affecting all matter equally, so there are no completely free particles, even in the deep sky. On the earth, experiences of USM can be carried out only in reduced regions of the space-time, for instance during a small enough duration or with objects moving without friction on an horizontal plane. The motion of a free particle is given by

$$r = r_0 + u\, t,$$

where the initial position $r_0 \in \mathbb{R}^3$ at $t = 0$ and the uniform velocity $u \in \mathbb{R}^3$ are constant. The event "the particle is passing through r_0 at $t = 0$" is represented in the considered coordinate system by

$$X_0 = \begin{pmatrix} 0 \\ r_0 \end{pmatrix} .$$

Introducing the 4-column

$$U = \begin{pmatrix} 1 \\ u \end{pmatrix} ,$$

the event "the particle is in r at t" is represented by

$$X = X_0 + U t \ . \tag{1.3}$$

Definition 1.10 *With respect to a given family of coordinate systems, a characteristic of an object or a quantity is* ***invariant*** *if its representation in all the systems of the family is identical. We talk also about the* ***invariance*** *of the caracteristic or the quantity.*

For instance, let us consider the family of the coordinate systems of observers for which the motion of the same particle is straight and uniform. We would like to ask the following question: which are the coordinate changes $X' \mapsto X$ of this family?

Theorem 1.11 *The coordinate changes for which are invariant:*

- *the uniform straight motions,*
- *the durations,*
- *the distances and angles,*
- *the oriented volumes,*

are regular affine maps of the following form:

$$X = PX' + C, \quad C = \begin{pmatrix} \tau \\ k \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \tag{1.4}$$

where $\tau \in \mathbb{R}$, $k \in \mathbb{R}^3$, $u \in \mathbb{R}^3$ and $R \in \mathbb{SO}(3)$ [Comment 1].

Proof. The parameterization 1.3 of the trajectory being affine, the coordinate change in $\mathbb{R}^4$ preserves straight lines and the middle of segments. As a parallelogram is a quadrilateral whose the diagonals meet in their middle, the coordinate change preserves parallelograms and, reasonning by reccurence, parallelepipeds and parallelotopes. So the coordinate change is affine:

$$X = PX' + C\ , \tag{1.5}$$

where $C \in \mathbb{R}^4$ and the 4×4 matrix P are constant. As the coordinate systems define one-to-one assignments from X into the event $\mathbf{X}$, the coordinate change is also one-to-one. Considering the difference of the columns representing two events $\mathbf{X}_1$ and $\mathbf{X}_2$ in the considered coordinate systems:

$$\Delta X = X_2 - X_1 = \begin{pmatrix} \Delta t \\ \Delta r \end{pmatrix}, \quad \Delta X' = X'_2 - X'_1 = \begin{pmatrix} \Delta t' \\ \Delta r' \end{pmatrix},$$

we obtain a linear relation:

$$\Delta X = P \, \Delta X'\ . \tag{1.6}$$

Next we put:

$$C = \begin{pmatrix} \tau \\ k \end{pmatrix}, \quad P = \begin{pmatrix} \alpha & w^T \\ u & F \end{pmatrix},$$

where $\alpha, \tau \in \mathbb{R}$, $u, w, k \in \mathbb{R}^3$ and F is a 3×3 matrix. Hence, 1.6 gives:

$$\Delta t = \alpha \, \Delta t' + w^T \Delta r' \ .$$

Identifying it with condition 1.1 ensuring the invariance of the duration gives:

$$\alpha = 1, \quad w = 0.$$

Hence, one has:

$$P = \begin{pmatrix} 1 & 0 \\ u & F \end{pmatrix} . \tag{1.7}$$

As P is regular, F must be so. Hence 1.6 gives for simultaneous events:

$$\Delta r = F \, \Delta r' \quad .$$

The invariance 1.2 of the distance reads:

$$(\Delta r')^T \, F^T \, F \, \Delta r' = (\Delta r')^T \, \Delta r'.$$

The column $\Delta r'$ being arbitrary, one obtains:

$$F^T \, F = 1_{\mathbb{R}^3} \ .$$

The matrix F is orthogonal. Taking into account that oriented volumes 1.8 are transformed as:

$$V' = \det(F) V \ ,$$

their invariance entails that F is a rotation that we denote R afterwards. As $\det(P) = \det(R) = 1$, P is regular and the affine map $X' \mapsto X$ so is. ■

Definition 1.12 *The coordinate changes 1.4 are called* ***Galilean transformations****. Any of them can be obtained composing elementary ones among:*

- ***clock change*** τ *(with* $k = u = 0,\ R = 0$*):* $t = t' + \tau$, $r = r'$,
- ***spatial translation*** k: $t = t'$, $r = r' + k$,
- ***rotation*** R: $t' = t$, $r = R\, r'$,
- ***Galilean boost*** *or* ***velocity of transport*** u: $t = t'$, $r = r' + ut$.

A general Galilean transformation reads:

$$r = R\, r' + u\, t' + k, \quad t = t' + \tau \ , \tag{1.8}$$

or in matrix form:

$$C = \begin{pmatrix} \tau \\ k \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix} \ , \tag{1.9}$$

1.3.2 Principle of relativity

If a particle is in Uniform Straight Motion for an observer, so is for any other observer. Hence all the coordinate systems in the sense defined by 1.6 are equivalent, including the ones in which the particle is at rest. In other words, we admit in particular the equivalence between the motion and rest. Galileo Galilei proposed in his famous "Dialogue concerning the two chief world systems" (1632) this point of view according to which the observations of physical phenomena do not allow to know if we are in motion or at rest, provided the motion is straight and uniform. **Galileo's principle of relativity** turns this from a negative to a positive statement:

Principle 1.13 *The statement of the physical laws of the classical mechanics is the same in all the coordinate systems in the sense defined by 1.6.*

For the moment, this principle is formulated in rather general words but we shall soon make it clearer in applications. By classical mechanics, let us recall that we consider phenomena for which the velocity of the light is so huge as it may be considered as infinite.

1.3.3 Space-time structure and velocity addition

Up to now, the space-time was a set of which the elements –the events– were parameterized by four coordinates. Considering only uniform straight motions, we need only affine transformations 1.5 for the coordinate changes. In other words, the space-time $\mathcal{U}$ may be perceived as an **affine space** of dimension 4 and the coordinates of an event $\mathbf{X}$ changes according to the transformation law for the component of one of its points. Hence, the structure of the space-time must not be imposed *a priori* but is deduced from the physical observations (the uniform straight motion).

Have a look now to our starting point, the uniform straight motion. In the old coordinate system, it reads:

$$X' = X'_0 + U't' \ .$$

Combining it with the Galilean transformation 1.4 gives:

$$X = P(X'_0 + U't') + C \ .$$

Accounting for 1.8, we recover 1.3, provided:

$$X_0 = P(X'_0 - U'\tau) + C \ , \tag{1.10}$$

$$U = PU' \ . \tag{1.11}$$

What do these relations know us?

- Without clock change, the first one reads:

$$X_0 = PX'_0 + C \ ,$$

that is nothing else the transformations law for the components of a point of $\mathcal{U}$. For more general transformations, the additional term in 1.10 takes into account the clock change.

- The second relation, 1.11, is the transformation law for the components of a vector $\vec{\mathbf{U}}$ of the vector space attached to $\mathcal{U}$. It will be called the 4-**velocity**.

Let us consider for instance a particle of velocity v' in the coordinate system X'. In another one X obtained from X' by a Galilean transformation 1.9, the 4-velocity is given by 1.11:

$$U = \frac{dX}{dt} = \begin{pmatrix} 1 \\ \dot{r} \end{pmatrix} = \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix} \begin{pmatrix} 1 \\ v' \end{pmatrix} , \tag{1.12}$$

Thus the velocity in the new coordinate system is:

$$\boxed{v = u + R\,v' \ .} \tag{1.13}$$

In particular, for a Galilean boost u, one has:

$$v = u + v' \ .$$

This is the **velocity addition formula**. Also, combining two Galilean boosts u_1 and u_2, we verify the resulting velocity of transport is:

$$u = u_1 + u_2 \ .$$

1.3.4 Organizing the calculus

For the convenience, an affine transformation $X' \mapsto X = PX' + C$ can be denoted $a = (C, P)$. Applying successively a_1 and a_2 gives a new affine transformation a_3:

$$a(X) = a_2(a_1(X)) = a_2(C_1 + P_1 X) = C_2 + P_2(C_1 + P_1 X) \ ,$$

hence:

$$a_3 = a_2 a_1 = (C_2, P_2)(C_1, P_1) = (C_2 + P_2 C_1, P_2 P_1) \ .$$

This product is associative and has an identity transformation $e = (0, 1_{\mathbb{R}^3})$ such as $e\,a = a\,e = a$. Each affine transformation $a = (C, P)$ has an inverse transformation $a^{-1} = (-P^{-1}C, P^{-1})$ such as $a^{-1}a = a\,a^{-1} = e$. It is straightforward to verify that the combination of two Galilean transformations a_2 and a_1 is also a Galilean transformation a given by:

$$u = u_2 + R_2 u_1, \quad R = R_2 R_1, \quad \tau = \tau_2 + \tau_1, \quad k = k_2 + R_2 k_1 + u_2 \tau_1 \ . \tag{1.14}$$

It is easy to verify that the inverse transformation $X \mapsto X' = P^{-1}X + C'$ is a Galilean transformation represented by [Comment 2]:

$$C' = \begin{pmatrix} \tau' \\ k' \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 \\ -R^T u & R^T \end{pmatrix} , \tag{1.15}$$

putting:

$$\tau' = -\tau, \qquad k' = -R^T(k - u\tau) \ .$$

It is often convenient to organize the matrix calculation by working rather in $\mathbb{R}^5$, representing the column X and the affine transformation $a = (C, P)$ respectively by:

$$\tilde{X} = \begin{pmatrix} 1 \\ X \end{pmatrix} \in \mathbb{R}^5 \qquad \tilde{P} = \begin{pmatrix} 1 & 0 \\ C & P \end{pmatrix} , \tag{1.16}$$

so the affine transformation 1.4 looks like a simple regular linear transformation:

$$\tilde{X} = \tilde{P}\tilde{X}' \ , \tag{1.17}$$

where, accounting for 1.9, the Galilean transformation a is represented by the 5×5 matrix decomposed by blocks:

$$\tilde{P} = \begin{pmatrix} 1 & 0 & 0 \\ \tau & 1 & 0 \\ k & u & R \end{pmatrix} \ . \tag{1.18}$$

In a similar way, accounting for 1.15, the inverse transformation is represented by:

$$\tilde{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \tau' & 1 & 0 \\ k' & -R^T u & R^T \end{pmatrix} \ . \tag{1.19}$$

1.3.5 About the units of measurement

The way is still long to cover the mechanics of continua but let us stop for a moment to have a look to the conversion of units. Let the event **X** be represented in a coordinate system by $\bar{X}$ where durations and times are measured with new units. Let say that the time and length units in the old coordinate system are equal respectively to T and L in the new one. The conversion of units is given by the scaling:

$$\bar{t} = T\ t, \qquad \bar{r} = Lr \ ,$$

or, in matrix form:

$$\bar{X} = P_u X \tag{1.20}$$

with:

$$P_u = \begin{pmatrix} T & 0 \\ 0 & L1_{\mathbb{R}^3} \end{pmatrix} \ . \tag{1.21}$$

Similarly, let us apply the scaling:

$$\bar{X}' = P_u X' \tag{1.22}$$

Combining the Galilean transformation 1.4 and the scalings 1.20 and 1.22 lead to:

$$\bar{X} = \bar{P}\bar{X}' + \bar{C} \ ,$$

with:

$$\bar{P} = P_u P P_u^{-1}, \qquad \bar{C} = P_u C \ .$$

Accounting for 1.9 and 1.21 shows that

$$\bar{C} = \begin{pmatrix} \bar{\tau} \\ \bar{k} \end{pmatrix}, \quad \bar{P} = \begin{pmatrix} 1 & 0 \\ \bar{u} & \bar{R} \end{pmatrix}, \tag{1.23}$$

with $\bar{C}$ being a simple scaling of C:

$$\bar{\tau} = T\ \tau, \qquad \bar{k} = Lk \ ,$$

and:

$$\bar{u} = (L/T)\ u, \qquad \bar{R} = R \ ,$$

As result of the conversion of units, the rotation is invariant while the boost u is scaled as a velocity. It is worth to observe that, in a conversion of units, a Galilean transformation $a = (C, P)$ turns into a Galilean transformation $\bar{a} = (\bar{C}, \bar{P})$ [Comment 3]. The conversion does not affect the Galilean feature of an affine transformation. Of course, calculations can be organized with 5×5 matrices:

$$\bar{\tilde{P}} = \tilde{P}_u \tilde{P} \tilde{P}_u^{-1} \qquad \text{where} \qquad \tilde{P}_u = \begin{pmatrix} 1 & 0 \\ 0 & P_u \end{pmatrix} \ .$$

1.4 Comments for experts

[Comment 1] This Theorem is related to the Toupinian structure of the space-time which gives a theoretical framework to the universal or absolute time and space.

[Comment 2] In fact, the set of all the Galilean transformation is a Lie group of dimension 10 called Galileo's group.

[Comment 3] Conversely, the normalizer of Galileo's group in the affine group is composed of the Galilean transformations themselves and the conversions of units 1.20.

Chapter 2

Statics

2.1 Introduction

In this chapter, the bodies occupy a volume or are pointwise (particles). The bodies can be subjected to various kinds of forces, for instance gravity, electromagnetic forces, contact forces. Our aim is to determine at which conditions a body subjected to several forces remains in uniform straight motion (USM). It the body occupy a volume, that means all its particles are in USM with the same velocity. Thus the body is at rest in some particular coordinate systems. Before proposing general conditions, we hope to develop some intuition by discussing simple situations in which –to make the understanding easier– the body is initially at rest for the observer. Of course, if the body remains at rest in the observer coordinate system, it is in USM in any other coordinate system resulting from a Galilean transformation.

Let us consider for instance a body immersed in a fluid. Under the effect of the gravity only, it moves downwards. In the other hand, according to Archimedes' principle, the buoyancy opposes to it by moving the body upwards. The body remains at rest when both opposite forces, gravity and buoyancy, are of same magnitude. Hence the balance of force is a necessary condition of static equilibrium but it is not sufficient, as shown in the next example.

Let us consider a weighing scale composed of two pans hung to the ends of a beam pivoting on the fulcrum at any given position. Anyway, the unknown weight on the left hand pan and the standard weight on the right hand one are equilibrated by the reaction of the fulcrum but, according to the position, the beam can nevertheless rotate. This is due to the moment of each force, *i.e.* its tendancy to rotate the object. The beam is at rest when the moments of both weights are opposite and of same magnitude.

2.2 Force torsor

2.2.1 Two dimensional model

The first step of our modeling is considering two dimensional systems as our weighing scale (figure 13.1). Let R_f be the fulcrum reaction, positive upwards, g be the gravity, m_u be the unkwown mass, at the distance d_u of the fulcrum, and m_s be the standard mass, at the distance d_s of the fulcrum. The vertical force resultant is:

$$F = R_f - (m_u + m_s)g \ .$$

The moment about a point is a scalar, its magnitude being the product of the force and the perpendicular distance between the point and the force axe, and its sign being positive if the force rotates the body clockwise and negatif if counterclockwise. The moment resultant about the fulcrum f is:

$$M = d_u m_u g - d_s m_s g \ .$$

The scale is at rest provided the both forces and moments are balanced:

$$F = 0, \qquad M = 0 \ . \tag{2.1}$$

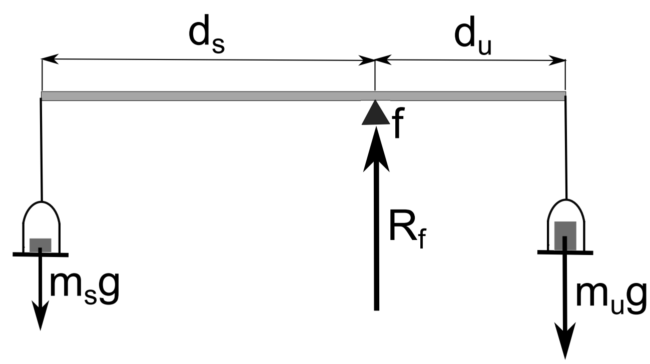

Figure 2.1: Weighting scale

We have calculated the moment about a particular point, the spulcrum. What happens if we calculate it about any given point on the beam, at the distance r_0 of the spulcrum (positive or negative according to this point being respectively at the left or the right of f):

$$M_r = (d_u - r_0)m_u g - (d_s + r_0)m_s g + rR_f \ ?$$

it is worth observing that:

$$M_{r_0} = M + Fr_0 \ .$$

This is the **transport law of the moment**. If the scale is at rest, the resultants F and M vanish, then M_{r_0} so is. The balance of moments occurs, whatever the position of the reference point is. The balance principle can read:

$$F = 0, \qquad M_{r_0} = 0 \ . \tag{2.2}$$

2.2.2 Three dimensional model

As second step, we generalize these simple ideas for three dimensional situations. Let a body be subjected to a system of forces represented by 3-columns $F_1, F_2, \cdots, F_N$, acting upon the body respectively at positions $r_1, r_2, \cdots, r_N$. The force resultant and the moment resultant about the origin of the coordinate system are:

$$F = \sum_{i=1}^{N} F_i, \qquad M = \sum_{i=1}^{N} r_i \times F_i \ .$$

The moment resultant about any other point of position r_0, taken as new origin, is:

$$M_{r_0} = \sum_{i=1}^{N} (r_i - r_0) \times F_i \ ,$$

hence:

$$M_{r_0} = \sum_{i=1}^{N} r_i \times F_i - r_0 \times \sum_{i=1}^{N} F_i = M - r_0 \times F \ ,$$

and, accounting for the anticommutativity of the cross product, the transport law of the moment reads:

$$M_{r_0} = M + F \times r_0 \ .$$

In three dimension, the balance principe equally reads 2.1 or 2.2, excepted that the scalars are replaced by 3-columns. Considering a coordinate systems for which the weighing balance is in the plan of r^1 and r^2 axis, it can be easily verified that the moment is directed along r^3 axis and to recover the previous expressions of M and M_{r_0}.

2.2.3 Force torsor and transport law of the moment

The third step of our modeling consists in constructing an object structured into force and moment components. As time is not concerned by the equilibrium of bodies, we work temporarily with the 4-column:

$$\breve{X} = \begin{pmatrix} 1 \\ r \end{pmatrix} \ ,$$

obtained by arising the second component of $\tilde{X}$, and the 4×4 matrix:

$$\breve{P} = \begin{pmatrix} 1 & 0 \\ k & R \end{pmatrix} \ ,$$

obtained by arising the second row and column of Galilean transformation (1.18), so the (**special**) **Euclidean transformation** $r = Rr' + k$ looks like a simple regular linear transformation:

$$\breve{X} = \breve{P} \breve{X}' \ . \tag{2.3}$$

Definition 2.1 *A **force torsor** $\breve{\mu}$ is an object represented in a coordinate system by a skew-symmetric 4×4 matrix:*

$$\breve{\mu} = \begin{pmatrix} 0 & F^T \\ -F & -j(M) \end{pmatrix} , \tag{2.4}$$

*where $F \in \mathbb{R}^3$ is its **force**, $M \in \mathbb{R}^3$ is its **moment** (or **torque**), and of which the components, under the Euclidean transformation 2.3, are modified according to the transformation law:*

$$\breve{\mu} = \breve{P}\breve{\mu}'\breve{P}^T . \tag{2.5}$$

Applying the rules of the matrix calculus, it is worth to notice that if $\breve{\mu}'$ is skew-symmetric, $\breve{\mu}$ given by 2.1 so is. Under an Euclidean transformation (and more generally under an affine transformation), the skew-symmetry property is preserved, that ensures the consistency of the definition [Comment 1]. To justify it in a physical point of view, we show first of all it allows to recover the transport law of the moment. By inversion of 2.5, one has:

$$\breve{\mu}' = \breve{P}^{-1}\breve{\mu}\breve{P}^{-T} , \tag{2.6}$$

To shift the origin at r_0 in the new coordinate system, let us consider a spatial translation $r' = r - r_0$. The coordinate change is represented by:

$$\breve{P}^{-1} = \begin{pmatrix} 1 & 0 \\ -r_0 & 1_{\mathbb{R}^3} \end{pmatrix} . \tag{2.7}$$

Introducing 2.4 and 2.7 into 2.6, the torsor is represented in the new coordinate system by:

$$\breve{\mu}' = \begin{pmatrix} 0 & F^T \\ -F & Fr_0^T - r_0F^T - j(M) \end{pmatrix} .$$

Accounting for 12.9 and the linearity of the map j, one has:

$$\breve{\mu}' = \begin{pmatrix} 0 & F^T \\ -F & -j(M + F \times r_0) \end{pmatrix} .$$

Thus, the force is not affected by the translation while the moment is transformed according to the **transport law of the moment** :

$$F' = F, \qquad M' = M + F \times r_0 . \tag{2.8}$$

In a similar manner, let us consider now a rotation $r' = R^T r$. The calculations are left to the reader and show that the force and moment rotate as the position:

$$F' = R^T F, \qquad M' = R^T M .$$

The translation and the rotation are particular cases of a general Euclidean transformation $r' = R^T(r-k)$. The coordinate change is represented by:

$$\breve{P}^{-1} = \begin{pmatrix} 1 & 0 \\ -R^T k & R^T \end{pmatrix} , \tag{2.9}$$

The matrix calculation 2.6 leads to the general transformation law:

$$\boxed{F' = R^T F, \qquad M' = R^T(M + F \times k) ,} \tag{2.10}$$

combining the transport and the rotation. It is easy to find two invariants under Euclidean transformations:

- the norm of the force: $\| F \|$ because of 12.17,
- the dot product of the force and the moment, owing to 12.17 and 12.18:

$$F' \cdot M' = (R^T F)^T R^T (M + F \times k) = F^T R R^T (M + F \times k) = F \cdot (M + F \times k) ,$$

$$F' \cdot M' = F \cdot M + F \cdot (F \times k) = F \cdot M + k \cdot (F \times F) = F \cdot M .$$

The linear space $\mathbb{M}_{44}^{skew}$ of the 4×4 skew-symmetric matrices is of dimension 6. Let $\mathbf{T}_s$ be the set of force torsors $\boldsymbol{\mu}$, in one-to-one correspondance with the skew-symmetric 4×4 matrices 2.4. Thanks to this map, $\mathbf{T}_s$ is a linear space of dimension 6 if we define by structure transport the addition of torsors and the multiplication of a torsor by a scalar.

2.3 Statics equilibrium

2.3.1 Resultant torsor

In a given coordinate system, let a body $\mathcal{B}$ be subjected to a system of forces represented $F_1, F_2, \cdots, F_N$, acting upon the body respectively at positions $r_1, r_2, \cdots, r_N$. The torsor of the force F_i about the origin of the coordinate system is:

$$\breve{\mu}_i = \begin{pmatrix} 0 & F_i^T \\ -F_i & -j(r_i \times F_i) \end{pmatrix} , \tag{2.11}$$

Definition 2.2 *The* ***net*** *or* ***resultant torsor*** *of a body $\mathcal{B}$ is the sum of the torsors of the forces acting upon it:*

$$\breve{\mu}(\mathcal{B}) = \sum_{i=1}^{N} \breve{\mu}_i .$$

If the resultant is null in a coordinate system, then it is so in any other coordinate system resulting from a Galilean transformation, because of 2.10.

2.3.2 Free body diagram and balance equation

Definition 2.3 *To identify the forces acting upon a body, it is convenient to draw a* ***free body diagram*** *, that is a sketch of the body and all* ***efforts*** *(forces or moments) acting upon it, by performing the three following steps:*

- *isolate the body,*
- *identify the forces and in particular, when removing all supports and connections, identify the corresponding* ***reactions****,*
- *make a sketch of the body, showing all forces acting on it.*

To illustrate step 2, let us consider usual kinds of supports:

- removing a **simple support**, we draw a force perpendicular to the surface on which the roller could roll,
- removing a frictionless **hinge**, we draw a force acting at the hinge centre,
- removing a **clamped** or **built-in support**, we draw a force acting unknown position near the support.

To solve a **statics problem**, we perform the following steps:

- draw a free body diagram,
- choose a convenient coordinate system to calculate the resultant moment,
- apply the **balance equation**:

$$\breve{\mu}(\mathcal{B}) = 0 \tag{2.12}$$

- solve it for the unknowns.

The resultant torsor having 6 scalar components, the number of balance equations is 6. Let m be the number of scalar unknowns (generally components of the support reactions). If $m = 6$, the body is said **isostatic**. Otherwise, the number of missing equations to solve the problem is $h = m - 6$ and is called the **redundancy degree**.

As example, let us consider an homogeneous rigid truss $\boldsymbol{PQ}$, of mass m and length L, hinged at $\boldsymbol{P}$ on an horizontal soil and supported at $\boldsymbol{Q}$ on a rough vertical wall, at the distance a of $\boldsymbol{P}$ (figure 2.2). The friction reaction at $\boldsymbol{Q}$ is tangential to the circle of centre $\boldsymbol{O}$ passing through $\boldsymbol{Q}$. To solve this statics problem, we draw the free body diagram to identify the forces acting on the truss (figure 2.3):

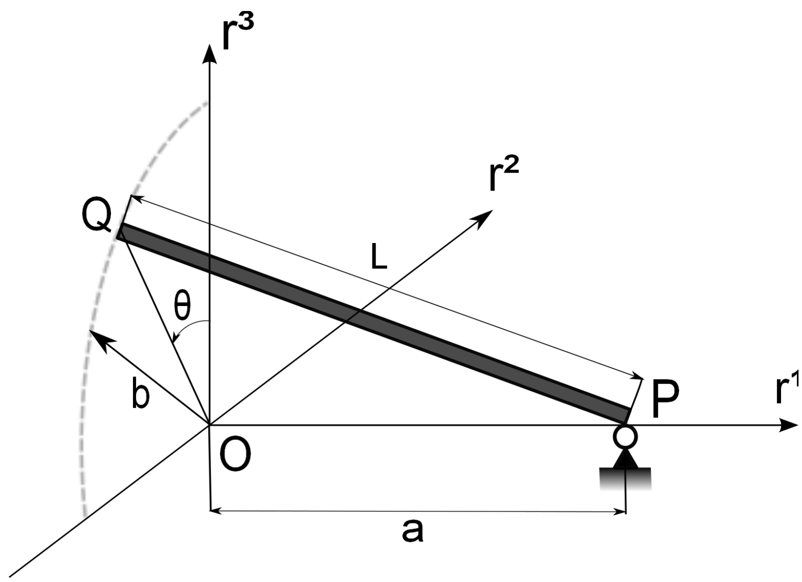

Figure 2.2: Rigid truss hinged on an horizontal soil and supported on a rough vertical wall

- its weight acting at the mass centre $\boldsymbol{G}$, middle of the truss:

$$W = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} ,$$

- the reaction acting at the hinge $\boldsymbol{P}$:

$$R_P = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} ,$$

- the reaction acting at $\boldsymbol{Q}$:

$$R_Q = \begin{pmatrix} R'_n \\ R'_t \cos\vartheta \\ R'_t \sin\vartheta \end{pmatrix} .$$

There are 6 unknowns: $R_1, R_2, R_3, R'_n, R'_t, \vartheta$. The problem is isostatic. The resultant torsor is null if its components so are. The force balance leads to:

$$R_1 = -R'_n, \qquad R_2 = -R'_t \cos\vartheta, \qquad R_3 = mg - R'_t \sin\vartheta ,$$

that allows to know the components R_1, R_2, R_3 of the reaction at $\boldsymbol{P}$, after the other unknowns has been determined. As the balance of moments occurs whatever is the

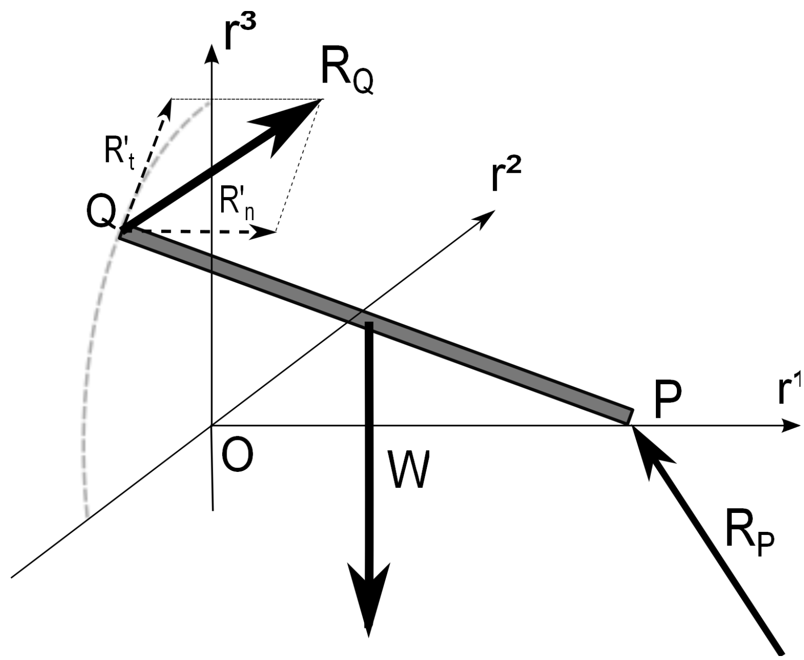

Figure 2.3: Free body diagram of the rigid truss

position of the reference point, it is wise to choose $\boldsymbol{P}$, that leads to:

$$M = \begin{pmatrix} -a \\ -b\sin\vartheta \\ b\cos\vartheta \end{pmatrix} \times \begin{pmatrix} R'_n \\ R'_t\cos\vartheta \\ R'_t\sin\vartheta \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -a \\ -b\sin\vartheta \\ b\cos\vartheta \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ,$$

thus:

$$-bR'_t + \frac{1}{2}mgb\sin\vartheta = 0, \qquad bR'_n\cos\vartheta + aR'_t\sin\vartheta - \frac{1}{2}mga = 0 ,$$
$$-aR'_t\cos\vartheta + bR'_n\sin\vartheta = 0 .$$

where $b = \sqrt{l^2 - a^2}$.

2.3.3 External and internal forces

If the body is in USM, all its part so are.

Definition 2.4 *We said that a body is in* ***static equilibrium*** *if the resultant torsor of each of its parts is null.*

This leads us to enforce the balance equations to any of its parts, provided we identify correctly the efforts acting upon it. For that, we make a cut throught the body to isolate the part. Next, we draw a free body diagram of the considered part in which –as we identify reactions when removing a support– we identified **internal forces** when cutting the body. For instance:

- cutting a **truss**, we draw a (tension or compression) force along the truss, away from the part,
- cutting a **cable**, we draw a tension force along the cable, away from the part,
- cutting a frictionless **pulley**, we draw a tension force along the cable on both side of the polley.

The other forces are called **external forces**.

Theorem 2.5 *The two following statements are equivalent:*

(i) the mutual forces of ***action and reaction*** *between complementary parts of a body are equal and opposite as well as their moments.*

(ii) The map $\mathcal{B} \rightarrow \breve{\boldsymbol{\mu}}(\mathcal{B})$ is an ***extensive quantity****, in the sense that:*

$$\breve{\boldsymbol{\mu}}(\mathcal{B}_1) + \breve{\boldsymbol{\mu}}(\mathcal{B}_2) = \breve{\boldsymbol{\mu}}(\mathcal{B}_1 \cup \mathcal{B}_2) \ ,$$

for any disjoint bodies $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proof. Indeed, let $\mathcal{A}$ a part of the body $\mathcal{B}$ and $\mathcal{A}'$ its complementary part. Let us split the exterior forces acting upon $\mathcal{B}$ into the ones acting on $\mathcal{A}$, of resultant torsor $\breve{\boldsymbol{\mu}}_A^{ext}$ and the ones acting on $\mathcal{A}'$, of resultant torsor $\breve{\boldsymbol{\mu}}_{A'}^{ext}$. Then, one has:

$$\breve{\boldsymbol{\mu}}(\mathcal{A} \cup \mathcal{A}') = \breve{\boldsymbol{\mu}}(\mathcal{B}) = \breve{\boldsymbol{\mu}}_A^{ext} + \breve{\boldsymbol{\mu}}_{A'}^{ext} \ .$$

On the other hand, $\breve{\boldsymbol{\mu}}_A^{int}$ being the resultant torsor of internal forces acting upon $\mathcal{A}$ and $\breve{\boldsymbol{\mu}}_{A'}^{int}$ being the one of internal forces acting upon $\mathcal{A}'$, it holds:

$$\breve{\boldsymbol{\mu}}(\mathcal{A}) = \breve{\boldsymbol{\mu}}_A^{ext} + \breve{\boldsymbol{\mu}}_A^{int} \ ,$$

$$\breve{\boldsymbol{\mu}}(\mathcal{A}') = \breve{\boldsymbol{\mu}}_{A'}^{ext} + \breve{\boldsymbol{\mu}}_{A'}^{int} \ .$$

From the three previous relations, it results:

$$\breve{\boldsymbol{\mu}}(\mathcal{A}) + \breve{\boldsymbol{\mu}}(\mathcal{A}') - \breve{\boldsymbol{\mu}}(\mathcal{A} \cup \mathcal{A}') = \breve{\boldsymbol{\mu}}_A^{int} \ + \breve{\boldsymbol{\mu}}_{A'}^{int}$$

if *(i)* holds, the right hand member is null, that proves *(ii)*. Conversely, if *(ii)* is true, the left hand member vanishes, that entails *(i)*. ■

In the sequel, we always shall suppose implicitly **Newton's third law**:

Law 2.6 *For any body in static equilibrium, the two equivalent statement are satisfied :*

- *the mutual forces of action and reaction between complementary parts of a body are equal and opposite as well as their moments,*

- *the resultant torsor is an extensive quantity.*

It is worth to observe that what we have done when we removed supports in the previous subsection is nothing else making a cut (mere semantic question). Indeed, let us call **foundation** the reunion of all the other bodies in the universe (although in fact only the vicinity of the support is relevant for us and we do not feel really concerned by what happens far away). Next we make the cut at the supports between the studied body and the foundation.

2.4 Comments for experts

[Comment 1]The torsor is in fact a skew-symmetric contravariant tensor of rank 2 and the corresponding transformation law for the components is 2.5.

Chapter 3

Dynamics of particles

3.1 Dynamical torsor

3.1.1 Transformation law and invariants

Definition 3.1 *In this chapter, a* ***particle*** *is some matter which is pointwise (for instance an elementary particle as an electron) or can be thought of as pointwise (for instance if it is seen from a long way off).*

In this chapter, we hope to model the motion of the particles. The torsor – introduced for the purpose of the Statics– is nothing else a 'prototype' of what we shall go doing throughout this book. Tackling the Dynamics is simply a matter of recovering an extra dimension, the time that we had provisionally arisen. Imitating the Statics model, we extend the notion of torsor in a space-time framework (for the moment, it is a simple game but it will take a strong meaning later on).

Definition 3.2 *The* ***dynamical torsor*** $\boldsymbol{\mu}$ *of a particle is an object represented in a coordinate system by a skew-symmetric* 5×5 *matrix:*

$$\boxed{\tilde{\mu} = \begin{pmatrix} 0 & T^T \\ -T & J \end{pmatrix} ,} \tag{3.1}$$

where $T \in \mathbb{R}^4$*,* $J \in \mathbb{M}_{44}^{skew}$ *and of which the components, under the Galilean transformation 1.17, are modified according to the transformation law:*

$$\tilde{\mu} = \tilde{P}\tilde{\mu}'\tilde{P}^T \ . \tag{3.2}$$

The linear space $\mathbb{M}_{55}^{skew}$ of the 5×5 skew-symmetric matrices is of dimension 10. Let $\mathbf{T}_d$ be the set of dynamical torsors $\boldsymbol{\mu}$, in one-to-one correspondance with the skew-symmetric 5×5 matrices 3.1. Thanks to this map, $\mathbf{T}_d$ is a linear space of dimension

10 if we define by structure transport the addition of torsors and the multiplication of a torsor by a scalar.

Taking into account the structure of the space-time, T and J are decomposed by blocks:

$$T = \begin{pmatrix} m \\ p \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -q^T \\ q & -j\,(l) \end{pmatrix}, \tag{3.3}$$

where m is scalar and $p, q, l \in \mathbb{R}^3$. Which is the physical meaning of these components? For this aim, we apply the transformation law of the torsor 3.2 or, equivalently, its inverse one:

$$\tilde{\mu}' = \tilde{P}^{-1} \tilde{\mu} \tilde{P}^{-T} \ . \tag{3.4}$$

Accounting for 1.19 and 3.3, the transformation law 3.4 itemizes in:

$$\boxed{m' = m \ ,} \tag{3.5}$$

$$\boxed{p' = R^T\,(p - m\,u) \ ,} \tag{3.6}$$

$$\boxed{q' = R^T\,(q - \tau'\,(p - m\,u)) + m'\,k' \ .} \tag{3.7}$$

$$\boxed{l' = R^T\,(l + u \times q) + k' \times (R^T(p - m\,u)) \ ,} \tag{3.8}$$

It is easy to spot expression 3.6 of p' in 3.8 and 3.7, then these two last relations alternatively read:

$$q' = R^T q + mk' - \tau' p' \ . \tag{3.9}$$

$$l' = R^T\,(l + u \times q) + k' \times p' \ , \tag{3.10}$$

To begin with, we observe the component m is invariant under any Galilean transformation, then fully characteristic of the particle. That aside, we admit these intricate expressions of the torsor components are rather puzzling. To see things clearly, we intend annihilating some of them by suitable Galilean transformations. For our aim, we discuss only the case that the invariant component m is not null. Starting in any coordinate system X, we choose the Galilean boost:

$$u = \frac{p}{m} \ , \tag{3.11}$$

which annihilates p' and reduces 3.9 to:

$$q' = R^T q + mk' \ .$$

Next, we pick the spatial translation:

$$k' = -\frac{1}{m}\ R^T q\ ,$$

which annihilates q'. As p' is null and with the boost 3.11, 3.10 is reduced to:

$$l' = R^T\,(l + u \times q) = R^T\,\left(l + \frac{1}{m}\ p \times q\right) = R^T\,\left(l - \frac{1}{m}\ q \times p)\right)\ . \tag{3.12}$$

There is nothing more to do because the change clock τ' occurs only in 3.9 but is multiplied by zero while the rotation R obviously cannot annihilate l'. To sum up, what is the result of this massacre? We killed the components p and q. It remains m, which is pleasantly invariant, and l.

Incidently, cast a glance to the 3-column occuring in the last relation:

$$l_0 = l - \frac{1}{m}\ q \times p\ . \tag{3.13}$$

Owing to 3.5, 3.10 and 3.9 and after obvious simplifications, this quantity becomes in any another coordinate system X':

$$l'_0 = l' - \frac{1}{m'}\ q' \times p' = R^T\,(l + u \times q) - \frac{1}{m}\ (R^T q) \times p'\ .$$

Accounting for 3.6 and 12.19, it holds:

$$l'_0 = R^T\,\left(l + u \times q - \frac{1}{m}\ q \times p + q \times u\right)\ ,$$

that leads to the transformation law:

$$\boxed{l'_0 = R^T l_0\ .} \tag{3.14}$$

A straightforward consequence is that the norm of l_0 is invariant. We can consider that a particle is characterized by two invariant quantities, m and $\parallel l_0 \parallel$.

3.1.2 Boost method

Conversely, let us consider a coordinate system X' in which the torsor has a **reduced form** (we have just finished proving the existence of such a coordinate system):

$$\tilde{\mu}' = \begin{pmatrix} 0 & T^T \\ -T & J \end{pmatrix} = \begin{pmatrix} 0 & m & 0 \\ -m & 0 & 0 \\ 0 & 0 & -j(l_0) \end{pmatrix}\ , \tag{3.15}$$

where there is no trouble to put m instead of m' because we know this component is invariant. We claim now the particle is at rest in this coordinate system, for convenience at the position $r' = 0$ at time $t' = 0$. We call **proper coordinate system of a particle** a coordinate system in which this particle is at rest at $X' = 0$. Of course, this proper coordinate system is not unique. Let $\bar{X}'$ another proper coordinate system of the particle. As $X' = 0$ must be transformed into $\bar{X}' = 0$, the changes $X' \mapsto \bar{X}'$ of proper coordinate systems are linear transformations.

What does the dynamical torsor tell us about a free particle in USM?

To know it, let us consider another coordinate system $X = PX' + C$ with a Galilean boost v (see Definition 1.12) and a translation of the origin at $k = r_0$ (hence $\tau = 0$ and $R = 1_{\mathbb{R}^3}$), providing the trajectory equation:

$$r = r_0 + vt \ . \tag{3.16}$$

of the particle moving in USM at velocity v. We can determine the new components of the torsor in X by performing the matrix product 3.2 applied to 3.15 or, alternatively, using formulae 3.5 to 3.7, providing:

$$p = m\,v, \qquad q = m\,r_0, \qquad l = l_0 + q \times v \ ,$$

or, taking into account the trajectory equation 3.16:

$$p = m\,v, \qquad q = m\,(r - vt), \qquad l = l_0 + r \times m\,v \ . \tag{3.17}$$

The last relation of 3.17 is called the **transport law of the angular momentum**. In fact, it is a particular case of the general transformation laws 3.8 and 3.10 when considering only a Galilean boost.

Thanks to our boost method, we obtained the torsor components in a consistent way revealing their physical meaning:

- We know by experience that the quantity of matter or **mass** –measured with a weighing scale– is independent of the choice of a coordinate system in the sense defined by 1.6, then we naturally identify it to m.

- The quantity p, proportional to the mass and to the velocity is called quantity of motion or **linear momentum**.

- The quantity q, proportional to the mass and to the initial position, provides the trajectory equation. It will be called **passage** because indicating the particle is passing through r_0 at time $t = 0$, although it could be called also quantity of position.

- The quantity l splits into two terms. The second one, $q \times v = r \times m\,v = r \times p$, is called **orbital angular momentum** to remind of the cross product traducing a small rotation. The first one, $l_0 = l - q \times p \,/\, m$, is called the **spin angular momentum** and its meaning will be discussed further. Their sum, l, is called the **angular momentum**, although it could be called also quantity of rotation.

The dynamic torsor, that was at the begining a mere intellectual speculation, have taken now a physical sense, leading us to name its components.

Definition 3.3 *The dynamical torsor is structured into two components:*

- *the **linear** 4**-momentum** T, itself substructured into:*
  - *the **mass** m,*
  - *the **linear momentum** p,*
- *and the **angular** 4**-momentum** J, itself substructured into:*
  - *the **passage** q,*
  - *the **angular momentum** l.*

The invariants of the dynamical torsor are:

- *the **mass** m,*
- *the **spin** $\parallel l_0 \parallel$.*

In matrix form, the dynamical torsor reads:

$$\boxed{\tilde{\mu} = \begin{pmatrix} 0 & m & p^T \\ -m & 0 & -q^T \\ -p & q & -j(l) \end{pmatrix} .} \tag{3.18}$$

What can we say about these components along the trajectory? We know by experience that the mass at rest is time independent. Moreover, the spin angular momentum l_0 being a characteristic at rest of the particle, it is natural to suppose it is also time independent (this intuition will be confirm latter on). In short, $\tilde{\mu}'$ is constant along the trajectory. Using derivative, it reads:

$$\dot{\tilde{\mu}}' = 0 \ .$$

It is worth to observe that, according to the transformation law 3.2 and because a Galilean transformation is time independent, it is true in any coordinate system, even if the particle is not at rest in it. On this ground, we claim that:

Law 3.4 *For a particle in USM, the 10 components of the dynamical torsor are constant along the trajectory, then **integrals of the motion**:*

$$\dot{\tilde{\mu}} = 0 \ . \tag{3.19}$$

We could find it also by remarking that, as m, l_0, u and r_0, the linear momentum $p = mv$, the passage $q = mr_0$ are time independent and so is the angular momentum $l = l_0 + q \times v$. Let us observe above all we have a first example of physical law obeying Galileo's principle of relativity 1.13.

3.2 Rigid body motions

3.2.1 Rotations

Before going further, we need to acquire skills managing rotations. Let $r' = R^T r$ be a coordinate change associated to a given rotation R. It is a a complex transformations and, to grasp it better, we break it into three rotations (figure 3.1):

- a rotation R_φ of angle φ about z's axis, bringing y's axis to $\bar{y}$'s axis, called **line of nodes**,
- a rotation R_ϑ of angle ϑ about the line of nodes, bringing the axis of $z = \bar{z}$ to the one of $\tilde{z}$,
- a rotation R_ψ of angle ψ about $\tilde{z}$'s axis.

The angles φ, ϑ, ψ are called **Euler's angles** and allow to describe any rotation by composing the three rotations:

$$r' = R_\psi^T \tilde{r} = R_\psi^T R_\vartheta^T \bar{r} = R_\psi^T R_\vartheta^T R_\varphi^T r \ ,$$

thus:

$$R = R_\varphi R_\vartheta R_\psi \ . \tag{3.20}$$

In details, one has:

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{3.21}$$

Now, we would like to study the infinitesimal rotations. Differentiating 12.18, it holds:

$$dR\, R^T = -R\,(dR)^T = -(dR\, R^T)^T \ , \tag{3.22}$$

hence this matrix is skew-symmetric. There exists $d\psi \in \mathbb{R}^3$ such that:

$$dR\, R^T = j(d\psi) \ ,$$

then:

$$dR = j(d\psi)\, R \ . \tag{3.23}$$

In particular, an infinitesimal rotation around the identity is:

$$dR = j(d\psi) \ . \tag{3.24}$$

Applying it to a vector $v \in \mathbb{R}^3$, one has:

$$dR\, v = d\psi \times v \ .$$

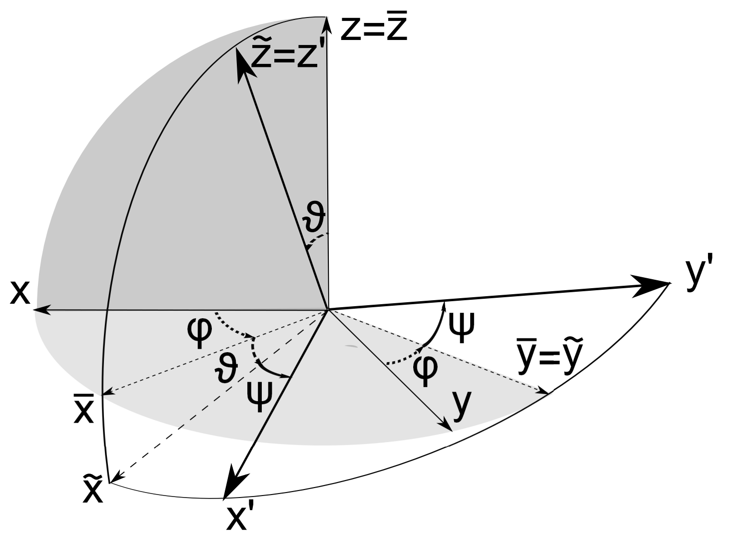

Figure 3.1: Euler's angles

An infinitesimal rotation is represented by a cross product and the axial vector $d\psi$ is called the **infinitesimal rotation vector**. Its direction provides the rotation axis and its norm measures the infinitesimal rotation angle. Alternatively, we can adopt the language of time derivatives instead of the one of differentials, dividing by dt in 3.23:

$$\dot{R} = j(\varpi)\, R \ , \tag{3.25}$$

where the axial vector $\varpi(t) = \dot{\psi}(t)$ is called **Poisson's vector**. Its direction provides the **instantaneous rotation axis** and its norm measures the **rotation rate**.

3.2.2 Rigid motions

We shall go reaching an important conceptual milestone by modeling arbitrary motions of rigid bodies (and not only USM as previously).

Definition 3.5 *A* ***rigid body*** *is such that all material lengths and angles remain unchanged by its motion. The motion of a rigid body is called a* ***rigid motion****.*

The transformations preserving the lengths and angles are Euclidean. Let X and X' be two coordinate systems in the sense defined by 1.6, X being arbitrary given while the particle is at rest in X'. Thus, at a given time t, the position r in X of any material particle of the particle with position r' in the system X' is given by an Euclidean transformation:

$$r = R(t)\, r' + r_0(t) \ . \tag{3.26}$$

In other words, the trajectory of this particle is modeled by the assignment:

$$t \mapsto r = R(t)\, r' + r_0(t) \ ,$$

describing the rigid motion. Hence this brings us to consider changes of coordinate systems of the space-time $X' \mapsto X$ composed of a rigid motion and a clock change:

$$r = R(t' + \tau)\, r' + r_0(t' + \tau), \qquad t = t' + \tau \ . \tag{3.27}$$

A Galilean transformation 1.8 is of the previous form with $r_0(t) = u\ t + k$ and a time independent rotation R but the changes of coordinate systems 3.27 are not in general Galilean transformations. In the sequel, we suppose that the maps $t \mapsto r_0(t)$ and $t \mapsto R(t)$ are smooth (continuously differentiable as far as needed by the calculations).

Now, we would like to discuss what is an infinitesimal rigid motion. Differentiating 3.27 and accounting for 3.25, one has:

$$dr = (\dot{r}_0 + \varpi \times (R r'))\, dt' + R\, dr', \qquad dt = dt' \ .$$

Eliminating r' thanks to 3.26, one has:

$$dr = (\dot{r}_0 + \varpi \times (r - r_0))\, dt' + R\, dr', \qquad dt = dt' \ ,$$

that can be recast in matrix form:

$$dX = P\, dX' \ , \tag{3.28}$$

where P is a linear Galilean transformation with the **velocity of transport**:

$$\boxed{u = \dot{r}_0(t) + \varpi(t) \times (r - r_0(t)) \ .} \tag{3.29}$$

If the changes of coordinate systems 3.27 are not in general Galilean transformations, infinitesimal such changes are linear Galilean transformations. For this reason, we introduce the following definition [Comment 1].

Definition 3.6 *The coordinate systems, in the sense defined by 1.6, which are deduced one from the other by changes 3.27 are called* ***Galilean coordinate systems****.*

In a practical point of view, the Galilean transformations can be used as approximation of 3.27, depending on the considered time and length scales at which we are working. Let us consider a particle of initial position $r_0 = 0$ and initial velocity u, moving with an uniform acceleration a. It is well known that the trajectory equation is:

$$r = u\, t + \frac{1}{2}\, a\, t^2 \ .$$

In the right hand member, the last term is negligeable with respect of the first one provided:

$$|\ t\ | \ll 2\, \frac{\|\ u\ \|}{\|\ a\ \|} \ .$$

During this time, the particle is almost gravitation free and we see, neglecting the last term, it covers a distance:

$$\mid r \mid \approx \parallel u \parallel \mid t \mid \ll 2 \, \frac{\parallel u \parallel^2}{\parallel a \parallel} \; .$$

This leads us to consider a 'space-time window' arround the initial event $X = 0$, of dimensions very small with respect to the previous time and distance thresholds, in which a particle is almost gravitation free and the changes of Galilean coordinates can be approximated by Galilean transformations [Comment 2].

3.3 Galilean gravitation

3.3.1 How to model the gravitational forces?

Gravitation is certainly not easy to describe in a consistent way but, as it is impossible to evade it, we decide to face up to it forthwith. We would like to generalize the law 3.4 of the dynamical torsor in the new context of the coordinate changes 3.27, according to Galileo's principle of relativity 1.13. Even if the torsor $\tilde{\mu}'$ is constant in some of them, it is not so in other ones, according to the transformation law 3.2 where the components of P –the rotation R and the velocity of transport 3.29– depend on time. Thus law 3.4 is only valid for the USM and cannot be generalized on its own to the Galilean coordinate systems.

Accounting for 1.16 and 3.1, the transformation law 3.2 itemizes in:

$$T = P\,T', \qquad J = P\,J'P^T + C(P\,T')^T - (P\,T')\,C^T \; . \tag{3.30}$$

In Definition 1.9, we presented the force as a phenomenon modifying the velocity of a particle. We would like now to give a more precise formulation in the context of rigid motions. As the linear momentum is the product of the mass by the velocity and the mass is constant, we claim the time derivative of the linear momentum is equal to the resultant force:

$$\dot{p} = F \; .$$

To develop some intuition of the suitable generalization, we consider a particle of mass m, at rest in some Galilean coordinate system X', hence $v' = 0$. For an observer turning at constant rotation velocity ϖ around some axis perpendicular to the plane containing both the observer and the particle, this last one moves in a plane along a circle, then it is deflected from the straight line, revealing the presence of a force. For the observer rotating at $r_0 = 0$ and working with a Galilean coordinate system X, the velocity of the particle v is given by the velocity addition formula 1.13 and the velocity of transport 3.29:

$$\dot{r} = v = u + v' = \varpi \times r \; . \tag{3.31}$$

Poisson's vector ϖ being constant, one has:

$$\dot{p} = m\dot{v} = m\varpi \times v = m\varpi \times (\varpi \times r) \ . \tag{3.32}$$

For convenience, let the plane be $r^1 r^2$, then $\varpi = \omega e_3$ and, for the observer, the particle is subjected to a force directed toward the axis :

$$\dot{p} = -m\omega^2 r \ . \tag{3.33}$$

This is the **centripetal force**, responsible for the observed deflection.

In terms of torsor component T, what have we done? We derivated the first relation of 3.30 with respect to time, accounting for T' being constant:

$$\dot{T} = \dot{P}\,T' \ ,$$

It can be easily checked we recover 3.33. This suggest forces can be generated by infinitesimal variation of Galilean transformations. On this ground, we invent a new way to derivate, accounting for the infinitesimal variation of P:

- firstly, we calculate the time derivative of $T = P\,T'$,
- next, we consider its limit as X' approaches X.

The result is denoted $\mathring{T}$ to distinguish it from the classical time derivative $\dot{T}$. The first step reads:

$$\frac{d}{dt}(P\,T') = P\,\dot{T}' + \dot{P}\,T' \ .$$

Next, when X' approaches X, T' approaches T and P approaches the identity:

$$\mathring{T} = \dot{T} + \dot{P}\,T \ .$$

Now, we generalize the law 3.4 by claiming that $\mathring{T}$ vanishes:

$$\dot{T} = -\dot{P}\,T \ .$$

The key-idea is to consider the right hand member models the gravitational forces.

3.3.2 Gravitation

Now we hope to provide a more precise formulation of this sketch. Provisionally, we swap the language of time derivatives for the one of differentials. If in each Galilean coordinate system some assignment $X \mapsto P(X)$ is given, by differentiating it with respect to X, we obtain a map

$$dX \mapsto dP = \Gamma(dX) \ ,$$

which is linear because linking infinitesimal quantities. Conversely, if such a map is given, does there exist a map $X \mapsto P(X)$ of which Γ is the differential? In fact, there is an hidden trap to avoid. Indeed, let $\mathbf{X}_i$ and $\mathbf{X}_f$ be two distinct events, represented in a Galilean coordinate system by X_i and X_f. Considering a path $\mathcal{C}_1$ from $\mathbf{X}_i$ to $\mathbf{X}_f$, one has:

$$X_i - X_f = \int_{\mathcal{C}_1} dX \ .$$

independently of the choice of the path. $\mathcal{C}_2$ being another path from $\mathbf{X}_i$ to $\mathbf{X}_f$, let us consider the loop $\mathcal{C}$ obtained by concatenation of $\mathcal{C}_1$ and $\mathcal{C}_2$. If the sense of $\mathcal{C}$ is the same as $\mathcal{C}_1$ and opposite to the one of $\mathcal{C}_2$, one has:

$$\oint_{\mathcal{C}} dX = \int_{\mathcal{C}_1} dX - \int_{\mathcal{C}_2} dX = 0 \ .$$

Considering another Galilean coordinate system X', we have to satisfy:

$$\oint_{\mathcal{C}'} P dX' = 0 \ . \tag{3.34}$$

$\mathcal{C}$ being the image of $\mathcal{C}'$ by the change of coordinate system $X' \mapsto X$.

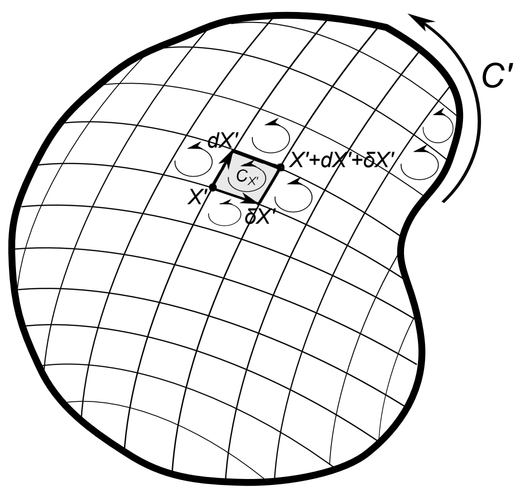

Figure 3.2: Mesh of elementary parallelograms

For practical purpose, let us consider a parallelogram $\mathcal{C}_{X'}$ of which the size approaches zero (figure 3.2). We can think in infinitesimal terms. Considering two distinct infinitesimal perturbations dX' and $\delta X'$ of X', the vertices of the elementary parallelogram are $X', X' + dX', X' + \delta X', X' + dX' + \delta X'$. Along the edge from X' to $X' + dX'$ of infinitesimal length, the transformation P is constant and equal to its value at X'. Along the edge from $X' + dX'$ to $X' + dX' + \delta X'$, the transformation is constant and equal to its value $P + dP$ at $X' + dX'$. Reasoning in a similar way for the two other edges, the condition 3.34 reads:

$$P dX' + (P + dP)\,\delta X' - (P \delta X' + (P + \delta P)\, dX') = 0 \ ,$$

hence, after simplification:

$$dP\,\delta X' - \delta P\,dX' = 0\ . \qquad (3.35)$$

Conversely, if this last condition is satisfied at any X and for any perturbations dX' and $\delta X'$, condition 3.34 is true for any loop $\mathcal{C}'$. Indeed, let us consider a surface of which the boundary is $\mathcal{C}'$ and let us mesh it with elementary parallelograms (figure 3.2). Adding up over all the meshes, the contributions over adjoining edges annihilate, the edges being run along twice in opposite sense. Only remains the contributions along the boundary $\mathcal{C}'$. The conditions 3.34 and 3.35 are equivalent but –from a practical viewpoint– the latter one is useful. Hence we claim that the map Γ must satisfy [Comment 3]:

$$\forall dX', \delta X', \qquad \Gamma(dX')\,\delta X' - \Gamma(\delta X')\,dX' = 0\ . \qquad (3.36)$$

3.3.3 Galilean gravitation and equation of motion

We define the **covariant differential** of T with respect to Γ as:

$$\mathbf{d}T = d(P\,T')|_{X'=X} = (P\,dT' + dP\,T')\,|_{X'=X} = (P\,dT' + \Gamma(dX)\,T')\,|_{X'=X}\ .$$

When X' approaches X, T' approaches T and P approaches the identity:

$$\mathbf{d}T = dT + \Gamma(dX)\,T\ . \qquad (3.37)$$

Theorem 3.7 *The maps $dX \mapsto dP = \Gamma(dX)$ of which the values are infinitesimal Galilean transformations and satisfying the condition 3.36 are of the following form:*

$$\boxed{\Gamma(dX) = \begin{pmatrix} 0 & 0 \\ j(\Omega)\,dr - g\,dt & j(\Omega)\,dt \end{pmatrix}\ ,} \qquad (3.38)$$

*where $g \in \mathbb{R}^3$ is called the **gravity** and $\Omega \in \mathbb{R}^3$ the **spinning**.*

Proof. Differentiating the expression of P in 1.9 and accounting for 3.24, an infinitesimal Galilean transformations around the identity reads:

$$dP = \begin{pmatrix} 0 & 0 \\ du & j(d\psi) \end{pmatrix}\ . \qquad (3.39)$$

where the 3-columns du and $d\psi$ linearly depend on dr and dt. Thus there exist 3×3 matrices A, B and 3-columns Ω, g such that:

$$d\psi = A\,dr + \Omega\,dt, \qquad du = B\,dr - g\,dt\ . \qquad (3.40)$$

Leaving out the primes, the condition 3.36 reads:

$$\begin{pmatrix} 0 & 0 \\ du & j(d\psi) \end{pmatrix} \begin{pmatrix} \delta t \\ \delta r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \delta u & j(\delta\psi) \end{pmatrix} \begin{pmatrix} dt \\ dr \end{pmatrix} ,$$

and is reduced to:

$$du\,\delta t - \delta u\,dt + j(d\psi)\,\delta r - j(\delta\psi)\,dr = 0 \ .$$

Introducing the expressions 3.40 into this last equation, we obtain after simplification and some simple algebraic manipulations:

$$(Tr(A)1_{\mathbb{R}^3} - A)^T dr \times \delta r + (B - j(\Omega))\,(\delta r\,dt - dr\,\delta t) = 0 \ .$$

The infinitesimal perturbations $dX, \delta X$ being arbitrary, it is satisfied if and only if:

$$Tr(A)1_{\mathbb{R}^3} = A, \qquad B = j(\Omega) \ . \tag{3.41}$$

The former equation is satisfied if and only if the matrix A is null. Introducing 3.40 into 3.39 and taking into account 3.41:

$$du = j(\Omega)\,dr - g\,dt, \qquad d\psi = \Omega\,dt \ , \tag{3.42}$$

achieves the proof. ■

Dividing both members of relation 3.37 by dt, accounting for the linearity of the map Γ and the definition 1.12 of the 4-velocity U, we define the **covariant derivative**:

$$\mathring{T} = \dot{T} + \Gamma(U)\,T \ . \tag{3.43}$$

Law 3.8 *For any particle only subjected to a* ***Galilean gravitation*** *3.38, the trajectory is governed by the equation:*

$$\mathring{T} = 0 \ .$$

Accounting for 1.12 and 3.38, one has:

$$\Gamma(U) = \begin{pmatrix} 0 & 0 \\ \Omega \times v - g & j(\Omega) \end{pmatrix} , \tag{3.44}$$

and the law 3.8 reads:

$$\begin{pmatrix} \dot{m} \\ \dot{p} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \Omega \times v - g & j(\Omega) \end{pmatrix} \begin{pmatrix} m \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

providing:

$$\dot{m} = 0, \qquad \dot{p} = m\,(g - 2\,\Omega \times v)\ . \tag{3.45}$$

The first equation means the mass is time independent along the trajectory. The second one gives the time rate of linear momentum in terms of the gravitation. Introducing the expression of p given by 3.17 into this equation and because the mass does not depends on time, we have:

$$m\dot{v} = m\,(g - 2\,\Omega \times v)\ , \tag{3.46}$$

that leads to **Souriau's equation of motion**:

$$\boxed{m\ddot{r} = m\,(g - 2\,\Omega \times v)\ ,} \tag{3.47}$$

allowing to determine the trajectory of the particle [Comment 4].

3.3.4 Transformation laws of the gravitation and acceleration

Before applying this law, we have still a few essential details to be settled. Indeed, we must not lose track of Galileo's principle of relativity 1.13. Guided by it, we claim this law is the same in all the Galilean coordinate systems. Thus in another Galilean coordinate system X', we must have:

$$\mathbf{d}T' = dT' + \Gamma'(dX')\,T'\ .$$

Introducing the first relation of 3.30 into 3.37, differentiating the products and taking into account 3.28 gives:

$$\mathbf{d}T = d(P\,T') + \Gamma(dX)\,P\,T' = P\,(dT' + (P^{-1}\Gamma(P\,dX')\,P + P^{-1}dP)\,T')\ ,$$

that is:

$$\mathbf{d}T = P\,\mathbf{d}T'\ , \tag{3.48}$$

provided the following **transformation law of the gravitation** is satisfied:

$$\Gamma'(dX') = P^{-1}(\Gamma(P\,dX')\,P + dP)\ . \tag{3.49}$$

Dividing both members of 5.46 by $dt = dt'$ leads to:

$$\mathring{T} = P\,\mathring{T}'\ . \tag{3.50}$$

As the matrix P is regular, the law 3.8 is valid in any Galilean coordinate system, then consistent with Galileo's principle of relativity. Dividing 3.49 by dt provides:

$$\Gamma'(U') = P^{-1}(\Gamma(U)\,P + \dot{P})\ .$$

Theorem 3.9 *In a Galilean coordinate change $X' \mapsto X$, a Galilean gravitation changes according to the transformation laws:*

$$\boxed{\Omega = R\,\Omega' - \varpi\ ,} \tag{3.51}$$

$$\boxed{g - 2\,\Omega \times v = a_t + R\,(g' - 2\,\Omega' \times v')\ .} \tag{3.52}$$

where:

$$\boxed{a_t = \dot{u} + \varpi \times (v - u)\ ,} \tag{3.53}$$

is called the ***acceleration of transport****.*

Proof. Indeed, owing to 1.9, 1.12, 1.15, 3.25 and 3.38, one has:

$$\Gamma' = \begin{pmatrix} 1 & 0 \\ -R^T u & R^T \end{pmatrix} \left(\begin{pmatrix} 0 & 0 \\ \Omega \times v - g & j(\Omega) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \dot{u} & j(\varpi)\,R \end{pmatrix} \right)\ .$$

Identifying the result of the calculation to the standard form 3.44 in the Galilean coordinate system X':

$$\Gamma'(U') = \begin{pmatrix} 0 & 0 \\ \Omega' \times v' - g' & j(\Omega') \end{pmatrix}\ ,$$

accounting for the linearity of j and 12.20, one has:

$$j(\Omega') = j(R^T(\Omega + \varpi)), \qquad \Omega' \times v' - g' = R^T(\Omega \times v - g + \dot{u} + \Omega \times u)\ . \tag{3.54}$$

As the map j is one-to-one, the first relation lead to the transformation law 3.51 for the spinning Ω. By obvious manipulations, the second relation o 3.54 reads:

$$g - \Omega \times v = \dot{u} + \Omega \times u + R\,(g' - \Omega' \times v')\ ,$$

We are now interested in the right hand side of the second equation 3.45. Owing to the following expression, it holds:

$$g - 2\,\Omega \times v = \dot{u} + \Omega \times (u - v) + R\,(g' - \Omega' \times v')\ .$$

According to 3.51, we have:

$$g - 2\,\Omega \times v = \dot{u} - \varpi \times (u - v) + (R\,\Omega') \times (u - v) + R\,(g' - \Omega' \times v')\ .$$

Accounting for 1.13 and 12.19, we transform the third term of the right hand side:

$$g - 2\,\Omega \times v = \dot{u} + \varpi \times (v - u) - R\,(\Omega' \times v') + R\,(g' - \Omega' \times v')\ .$$

We obtain the transformation law 3.52 for the right hand side of the second equation 3.45. ■

Concerning the previous theorem, we would like to make two comments about the terminology:

- the reason why Ω is called the spinning is the corresponding transformation law 3.51, taking into account ϖ represents a time rate of rotation.
- a_t is called acceleration of transport because, substituting u to r_0 and v to r in 3.29, it is the analoguous of the velocity of transport.

Before going further, we wish for checking both members of the second equation 3.45 are identically transformed under any change of Galilean coordinate system $X \mapsto X'$. Accounting for 3.46, we have:

$$\dot{v} = g - 2\,\Omega \times v \ .$$

Time derivating both members of 1.13 and taking into account 3.25 provides:

$$\dot{v} = \dot{u} + j(\varpi)\, R\, v' + R\, \dot{v}' = \dot{u} + \varpi \times (R\, v') + R\, \dot{v}' \ .$$

Accounting for 1.13, we transform the second term, that leads to the **transformation law of the acceleration**:

$$\boxed{\dot{v} = a_t + R\, \dot{v}' \ ,} \tag{3.55}$$

which fits 3.52.

Next, we wish for discussing the structure of the acceleration of transport. Time derivating the expression 3.29 of the velocity of transport and introducing it into 3.53 provides:

$$a_t = \ddot{r}_0 + \dot{\varpi} \times (r - r_0) + \varpi \times (\dot{r} - \dot{r}_0) + \varpi \times (v - u) \ , \tag{3.56}$$

in which we eliminate $\dot{r}_0$ thanks to 3.29 :

$$\dot{r} - \dot{r}_0 = v - u + \varpi \times (r - r_0) \ ,$$

that leads to the **decomposition of the acceleration of transport** into four terms:

$$\boxed{a_t = \ddot{r}_0 + \dot{\varpi} \times (r - r_0) + \varpi \times (\varpi \times (r - r_0)) + 2\varpi \times (R\, v') \ ,} \tag{3.57}$$

also, accounting for 12.14, equal to:

$$a_t = \ddot{r}_0 + \dot{\varpi} \times (r - r_0) + (\varpi \cdot (r - r_0))\varpi - \parallel \varpi \parallel^2 (r - r_0) + 2\varpi \times (R\, v') \ .$$

These expressions are general and allow for instance to recover the centripetal force of subsection 3.3.1 by taking $v' = r_0 = 0$ and considering ϖ is time independent. Throughout this book, we conform to the standard terminology with a few exceptions. It is one of them, the acceleration of transport referring in the literature only to the three first terms of 3.57.

It is worth to remark that Theorem 3.9 gives the transformation law of Ω and $g - 2\,\Omega \times v$ but not directly of the gravity g. Let us provide a transformation law depending only on the event (through r and t) and not on the velocity v, *i.e.* depending also on the neighbour events on the trajectory. Taking into account 1.13, 3.51 and 12.19, the transformation law 3.52 becomes:

$$g - 2\,\Omega \times v = a_t + R\,g' - 2\,(R\Omega') \times (Rv') = a_t + R\,g' - 2\,(\Omega + \varpi) \times (v - u)\ , \quad (3.58)$$

Let us remark that the acceleration of transport 3.56 is an affine function of v:

$$a_t = a_t^* - \varpi \times u + 2\,\varpi \times v\ , \qquad (3.59)$$

where ϖ, u and:

$$a_t^* = \ddot{r}_0 + \dot{\varpi} \times (r - r_0) - \varpi \times \dot{r}_0\ , \qquad (3.60)$$

depends on r and t but not on v. Introducing 3.59 into 3.58 leads to:

$$g = a_t^* + \varpi \times u + 2\,\Omega \times u + R\,g'\ . \qquad (3.61)$$

Eliminating $\dot{r}_0$ in 3.60 thanks to 3.29 and puting the expression of a_t^* into the previous relation provides the **transformation law of the gravity** :

$$\boxed{g = \ddot{r}_0 + \dot{\varpi} \times (r - r_0) + \varpi \times (\varpi \times (r - r_0)) + 2\,\Omega \times u + R\,g'\ ,} \qquad (3.62)$$

where the spining Ω is given by 3.51. Hence, if g' and Ω' depend only on the event through r' and t', g and Ω are explicit functions of r and t thanks to 3.27, 3.51 and 3.62.

Before going further, let us have a look once again to the exemple of subsection 3.3.1. Let us consider some Galilean coordinate system X' such that $g' = \Omega' = 0$. In absence of other force, a particle initialy at rest remains at rest later on. For an observer turning at constant rotation velocity and working with the coordinate system X, one has:

$$r = R(t)r', \qquad r_0 = 0, \qquad \dot{\varpi} = 0\ , \qquad (3.63)$$

and 3.31. Hence 3.51 and 3.62 give:

$$\Omega = -\varpi, \qquad g = -\varpi \times (\varpi \times r)\ , \qquad (3.64)$$

and the equation 3.46 of motion allows recovering 3.32:

$$m\dot{v} = m\,(g - 2\,\Omega \times v) = m\,(-\varpi \times (\varpi \times r) + 2\,\varpi \times (\varpi \times r)) = m\,\varpi \times (\varpi \times r)\ .$$

In this example, it is worth to remark that the second term envolving Ω is the double of the gravity g and, in general, it cannot be considered small or negligeable.

3.4 Newtonian gravitation

Among all the Galilean connections, there exists only one corresponding to our physical world. In classical mechanics where the velocity of the light is infinite, we can state **Newton's law of gravitation**:

Law 3.10 *There exists particular Galilean coordinate systems, called **inertial** or **Newtonian coordinate systems**, for which ones the gravitation resulting from a particle of mass m' passing through r' at t is given by:*

$$g = -\frac{k_g\, m'}{\| r - r' \|^2}\, \frac{r - r'}{\| r - r' \|}, \qquad \Omega = 0 \ , \tag{3.65}$$

*where k_g is the **gravitational constant**, equal to $6,674 \cdot 10^{-11} Nm^2 kg^{-2}$. We call it a **Newtonian gravitation** [Comment 5].*

Using the transformation law 3.51 and 3.52 allows to determine the expression of the gravitation in any other Galilean coordinate system, where –it is worth to remark it– the spinning Ω is not in general null.

We would like to determine the motion of a spinless particle of mass m arround the mass m' passing, for convenience, through $r' = 0$ at every t in some Newtonian coordinate system, governed by the law 3.8 then the equation of motion 3.47:

$$m\ddot{r} = m\, g = -m\mu \frac{r}{\| r \|^3} \ . \tag{3.66}$$

where $\mu = k_g m'$. The particle being spinless and accounting for 3.65, the time rate of the angular momentum given by 3.17 is:

$$\dot{l} = \frac{d}{dt}(r \times mv) = mv \times v + r \times mg = 0 \ , \tag{3.67}$$

because g is collinear to r. We discovered an integral of the motion. The information is unvaluable. Indeed, we have:

$$r \cdot l = r \cdot (r \times mv) = 0 \ ,$$

hence r lies in the plane orthogonal to the constant angular momentum. Picking z's axis along l and working in polar coordinates (ϱ, ϑ, z), one has:

$$r = \begin{pmatrix} \varrho \cos\vartheta \\ \varrho \sin\vartheta \\ 0 \end{pmatrix} , \qquad v = \dot{r} = \begin{pmatrix} \dot{\varrho}\cos\vartheta - \varrho\dot{\vartheta}\sin\vartheta \\ \dot{\varrho}\sin\vartheta + \varrho\dot{\vartheta}\cos\vartheta \\ 0 \end{pmatrix} ,$$

$$\ddot{r} = \begin{pmatrix} \ddot{\varrho}\cos\vartheta - 2\dot{\varrho}\dot{\vartheta}\sin\vartheta - \varrho\ddot{\vartheta}\sin\vartheta - \varrho\dot{\vartheta}^2\cos\vartheta \\ \ddot{\varrho}\sin\vartheta + 2\dot{\varrho}\dot{\vartheta}\cos\vartheta + \varrho\ddot{\vartheta}\cos\vartheta - \varrho\dot{\vartheta}^2\sin\vartheta \\ 0 \end{pmatrix} .$$

Performing the dot product of both members of equation 3.66 by $e_\varrho = r \;/\; \parallel r \parallel$ leads to:

$$\ddot{\varrho} - \varrho\dot{\vartheta}^2 = -\frac{\mu}{\varrho^2} \; . \tag{3.68}$$

Also, the angular momentum:

$$l = mr \times v = \begin{pmatrix} 0 \\ 0 \\ m\varrho^2\dot{\vartheta} \end{pmatrix} ,$$

is time independent, thus:

$$\dot{\vartheta} = \frac{h}{\varrho^2},$$

where h is a constant. Introducing the previous expression into 3.68 leads to:

$$\ddot{\varrho} - \frac{h^2}{\varrho^3} = -\frac{\mu}{\varrho^2} \; . \tag{3.69}$$

With the shrewd choice of the new variable $u = 1 \;/\; \varrho$, we obtain the linear differential equation:

$$\frac{d^2u}{d\vartheta^2} + u = \frac{\mu}{h^2} \; ,$$

or, introducing $y = u - \mu/h^2$:

$$\frac{d^2y}{d\vartheta^2} + y = 0 \; .$$

Multiplying by $dy/d\vartheta$ leads to:

$$\frac{dy}{d\vartheta}\frac{d^2y}{d\vartheta^2} + y\,\frac{dy}{d\vartheta} = \frac{1}{2}\frac{d}{d\vartheta}\left(\left(\frac{dy}{d\vartheta}\right)^2 + y^2\right) = 0 \; .$$

Hence, there exists a constant C such that:

$$\left(\frac{dy}{d\vartheta}\right)^2 + y^2 = C \; .$$

By integration we obtain:

$$\vartheta = \int \frac{dy}{\sqrt{C^2 - y^2}} + \vartheta_0 = \arcsin\left(\frac{y}{C}\right) + \vartheta_0 \; ,$$

and by inversion:

$$y = C\sin(\vartheta - \vartheta_0) \; .$$

We can make $\vartheta_0 = \pi/2$ by choice of the line $\vartheta = 0$. Then, coming back to the variable u, one has:

$$u = \frac{\mu}{h^2} - C\,\cos\vartheta \; ,$$

and the equation of the trajectory reads:

$$u = \frac{1}{\varrho} = \frac{\mu}{h^2}\,(1 + \varepsilon\,\cos\vartheta)\ . \tag{3.70}$$

The trajectory is a conic section of eccentricity ε, the mass m' being situated at a focus of the conic. It is an ellipse, parabola or hyperbola, according as $\varepsilon < 1$, $\varepsilon = 1$, or $\varepsilon > 1$. The motion is completely determined and fits the empirical laws discovered by Kepler (between 1605 and 1618).

It is worth to notice the decisive part played by the angular momentum, one of the torsor components, in solving the problem. In the concerned problem, the angular momentum remains an integral of the motion because the Newtonian gravitation 3.65 generates a **central force**. Unfortunately, conversely to what happens in USM, the torsor components are not yet in general integrals of the motion, excepted by chance as it occured now for the angular momentum. Nevertheless, other **integrals of the motion** than the torsor components can be found:

- **The energy.** The Newtonian gravitation is such that:

  $$g \cdot v = -\dot{\phi}\ , \tag{3.71}$$

  where:

  $$\phi = -\frac{\mu}{\parallel r - r' \parallel}\ , \tag{3.72}$$

  is called the **gravitation potential**. Introducing the **kinetic energy**:

  $$e = \frac{1}{2}\,m \parallel v \parallel^2\ , \tag{3.73}$$

  we verify the **total energy**:

  $$e_T = e + m\phi\ , \tag{3.74}$$

  is an integral of the motion because of the equation of motion 3.66:

  $$\dot{e}_T = m\ddot{r} \cdot v - mg \cdot v = 0\ .$$

- **Laplace-Runge-Lenz vector.** It is defined as:

  $$w_L = v \times l + m\,\phi\,r\ .$$

  Because of the angular momentum conservation 3.67, one has:

  $$\dot{w}_L = \dot{v} \times l + m\,\dot{\phi}\,r + m\,\phi\,v\ .$$

  For a spinless particle, the expressions 3.17 of the momenta and 3.66 give:

  $$\dot{w}_L = g \times (r \times mv) + m\,\dot{\phi}\,r + \phi\,p\ .$$

using the vector triple product 12.14, we obtain:

$$\dot{w}_L = (g \cdot v + \dot{\phi})\, mr + (\phi - g \cdot r)\, p \ .$$

Owing to 3.71 and verifying from 3.65 and 3.72 that the second parenthesis is null, we prove the Laplace-Runge-Lenz vector is an integral of the motion. In general, the integrals of the motion are powerful tools to determine the trajectory. For instance in Kepler's problem, using both the integrals of the motion l and w_L allows to show geometrically the trajectory is a conic section without integrating the differential equation 3.69. Indeed, we know from the conservation of the angular momentum that the trajectory lies in the plane orthogonal to l. Next, let us observe that:

$$w_L \cdot r = (r \times v) \cdot l + m\, \phi\, \varrho^2 \ ,$$

hence, one has:

$$\| \, w_L \, \| \ \varrho \cos \vartheta = \frac{1}{m} \ \| \, l \, \|^2 - m \mu \varrho \ ,$$

$$(m\, \mu + \| \, w_L \, \| \ \cos \vartheta)\, \varrho = m\, h^2 \ .$$

which leads to the equation 3.70 of a conic section with the eccentricity:

$$\varepsilon = \frac{\| \, w_L \, \|}{m\, \mu} \ .$$

3.5 Other forces

3.5.1 General equation of motion

The gravitation forces are odd, reason for which we have set aside a particular presentation for them. The other ones are very miscellaneous. We already know the reaction forces at a simple support and the internal forces identified when drawing a free body diagram, but there are many other ones, for instance of electromagnetic or chemical origin. As far as we are concerned here, we only define the minimal properties expected from these forces. Introducing a 4-column:

$$H = \begin{pmatrix} 0 \\ F \end{pmatrix} \ , \tag{3.75}$$

where the component $F \in \mathbb{R}^3$ represents the **resultant of the other forces**, we can generalize the law 3.8 as follows:

Law 3.11 *For any particle subjected to a Galilean gravitation and other forces, the trajectory is governed by the equation:*

$$\mathring{T} = H \ .$$

To be consistent with Galileo's principle of relativity 1.13, accounting for 3.50, H must be transformed under a change of Galilean coordinate systems $X' \mapsto X$ according to the transformation law of a 4-vector:

$$H = P\,H' \;, \tag{3.76}$$

which, owing to 1.9, leaves the first component null while we recover the transformation law 2.10 of the force component F. In details, the equations 3.45 are generalized in presence of other forces:

$$\boxed{\dot{m} = 0, \qquad \dot{p} = m\,(g - 2\,\Omega \times v) + F \;,} \tag{3.77}$$

or in short:

$$\dot{p} = F_\Gamma + F \;,$$

where the **gravitation force**:

$$F_\Gamma = m\,(g - 2\,\Omega \times v) = F_g \;\; - F_C \;,$$

is decomposed as the difference of:

- the **gravity force** $F_g = m\,g$,
- and the **Coriolis force** $F_C = 2\,m\,\Omega \times v$.

Nevertheless, we should be careful about these notations because the nature and the transformation laws of F_Γ and F are completely different.

3.5.2 Foucault's pendulum

We would like to determine the motion of a pendulum at the latitude λ in northern hemisphere (figure 3.3). The bob, suspended from a point $\boldsymbol{P}$ by a light thread of length l, can be considered as a particle of mass m. To identify the forces, we draw a free body diagram of the bob. Cutting the thread, we consider the tension force F of intensity S along the thread, away from the bob. At this scale, it is reasonnable to consider a constant gravity g, directed toward the earth's centre. Let us pick a Galilean coordinate system with $\boldsymbol{O}z$ directed vertically upward (as determined by a plumb line) passing through the suspension $\boldsymbol{P}$ at $z = l$, and $\boldsymbol{O}x$ pointing south. In the absence of more accurate information, the spinning Ω is assumed uniform with intensity $\Omega_\oplus$ and directed along earth's axis:

$$\Omega = \Omega_\oplus \begin{pmatrix} -\cos\lambda \\ 0 \\ \sin\lambda \end{pmatrix} \;. \tag{3.78}$$

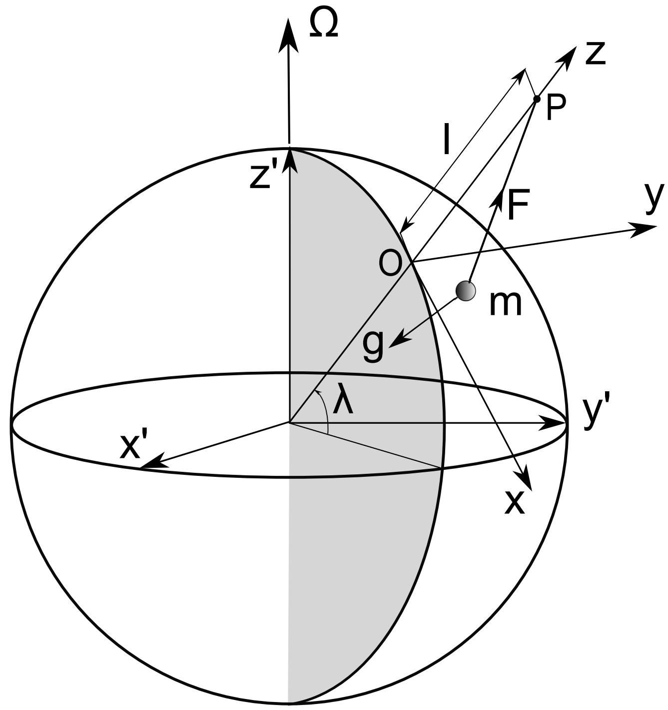

Figure 3.3: Foucault's pendulum

The equation 3.77 reads:

$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = m\ \parallel g \parallel \begin{pmatrix}0\\0\\-1\end{pmatrix} - 2m\Omega_{\oplus}\begin{pmatrix}-\cos\lambda\\0\\ \sin\lambda\end{pmatrix} \times \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} + \frac{S}{l}\begin{pmatrix}-x\\-y\\l-z\end{pmatrix} ,$$

leading to:

$$m\ddot{x} = 2m\Omega_{\oplus}\dot{y}\sin\lambda - \frac{S}{l}x \ , \tag{3.79}$$

$$m\ddot{y} = -2m\Omega_{\oplus}\left(\dot{x}\sin\lambda + \dot{z}\cos\lambda\right) - \frac{S}{l}y \ , \tag{3.80}$$

$$m\ddot{z} = -m\ \parallel g \parallel + 2m\Omega_{\oplus}\dot{y}\cos\lambda + \frac{S}{l}(l-z) \ , \tag{3.81}$$

The bob moves on the sphere of centre $\boldsymbol{P}$, radius l and equation:

$$x^2 + y^2 + (z-l)^2 - l^2 = 0$$

For small disturbances x, y and z, neglecting z with respect to l:

$$z = \frac{1}{2l}\left(x^2 + y^2\right) ,$$

showing that if x, y are small of the first order, z is small of the second order. Neglecting the terms containing z and $\ddot{z}$, equation 3.81 degenerates into:

$$S = m \parallel g \parallel -2m\Omega_\oplus \dot{y} \cos\lambda \ .$$

Introducing this expression into former equations 3.79, 3.80 and neglecting terms containing small quantities of second order, $\dot{y}x$, $\dot{y}y$ and $\dot{z}$, we obtain:

$$\ddot{x} - 2\Omega_\oplus \dot{y} \sin\lambda + \omega^2 x = 0 \ ,$$

$$\ddot{y} - 2\Omega_\oplus \dot{x} \sin\lambda + \omega^2 y = 0 \ ,$$

where $\omega = \sqrt{\parallel g \parallel \ / \ l}$. Introducing $\zeta = x + i\, y$, the pair of equation reads:

$$\ddot{\zeta} + 2i\Omega_\oplus \dot{\zeta} \sin\lambda + \omega^2 \zeta = 0 \ ,$$

and, neglecting $\Omega_\oplus^2$ with respect to ω^2, the general solution is:

$$\zeta = (A\, e^{i\omega t} + B\, e^{-i\omega t})\, e^{-i\Omega_\oplus t \sin\lambda} \ .$$

The first factor on the right represents an elliptic motion. The effect of the second factor is to make this ellipse rotate with angular velocity $-\Omega_\oplus \sin\lambda$, which is clockwize in the northern hemisphere and counterclockwize in the southern one.

The first observation of this slow shift of the ellipse was made by the physicist Léon Foucault in 1851. The usual interpretation of this experiment is that it allows to observe "in laboratory" earth's rotation about its axis and to measure the corresponding circular frequency $\omega_\oplus$ [Comment 6]. Indeed, let X' be a coordinate system with the space origin at earth's centre $\boldsymbol{O}$ and $\boldsymbol{O}z'$ chosen as earth's rotation axis. Assuming this coordinate system X' is Newtonian, one has: $\Omega' = 0$. Swaping X' for X and using the spinning transformation law 3.51, one has:

$$\Omega = R^T \varpi \ , \tag{3.82}$$

The rotation being described by Euler's angles $\varphi = \omega_\oplus t$, $\vartheta = \pi/2 - \lambda$ and $\psi = 0$, formula 3.21 provides:

$$R = \begin{pmatrix} \sin\lambda\cos(\omega_\oplus t) & -\sin(\omega_\oplus t) & \cos\lambda\cos(\omega_\oplus t) \\ \sin\lambda\sin(\omega_\oplus t) & \cos(\omega_\oplus t) & \cos\lambda\sin(\omega_\oplus t) \\ -\cos\lambda & 0 & \sin\lambda \end{pmatrix} \ ,$$

which entails:

$$\dot{R}R^T = \omega_\oplus \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

Using 3.25 –which reads also $\dot{R}R^T = j(\varpi)$– and 12.8, we deduce Poisson's vector:

$$\varpi = \begin{pmatrix} 0 \\ 0 \\ \omega_\oplus \end{pmatrix}$$

that allows recovering the spinning 3.78 of the "laboratory" by 3.82, provided the spinning frequency $\Omega_\oplus$ is equal to earth's rotation $\omega_\oplus$.

The experience was realized at the Pantheon in Paris thanks to a $67\,m$ pendulum. With an oscillation amplitude of $3\,m$, the thread deviates of $2°33'$ from the plumb line and the bob rises up $z = 6,7\,cm$, that is 10^{-3} of the thread length l, justifying the above approximation of small perturbations. The proper circular frequency of the pendulum is $\omega = \sqrt{9,81/67} = 0,382\,rd/s$ and its period is $T_p = 2\,\pi/\omega = 16,4\,s$. The circular frequency of the ellipse rotation at Paris latitude $\lambda = 48°51'24''$ is $\Omega_\oplus \sin\lambda = 5,49 \cdot 10^{-5}\,rad/s$ and the period is $31h\,46'50''$. The rotation velocity of the ellipse is $11,32°$. Thus earth's rotation was observable but the accuracy was not better than a few percent. The difference between the sideral day of $23h\,56'04''$ and the normal $24h$ day is so small (namely $0,27\%$) that the distinction was (and is still now) beyond observation.

3.5.3 Thrust

So far we have assumed the mass was constant but this hypothesis is not very necessary and we can model within the present framework objects with variable mass. When a body such a rocket expels mass in a direction, this mass will cause a force of equal magnitude but opposite direction called a **thrust**. Let w be the velocity of the exhaust gases with respect to a Galilean coordinate system X' in which the rocket of mass m is at rest. The thrust is modeled by a the 4-column:

$$H' = \dot{m} \begin{pmatrix} 1 \\ w \end{pmatrix} . \tag{3.83}$$

In a coordinate system X obtained from X' by a boost v, the rocket has a velocity v and the thrust is given by its transformation law 3.76:

$$H = \begin{pmatrix} 1 & 0 \\ v & 1_{\mathbb{R}^3} \end{pmatrix} \dot{m} \begin{pmatrix} 1 \\ w \end{pmatrix} = \dot{m} \begin{pmatrix} 1 \\ v + w \end{pmatrix} . \tag{3.84}$$

Accounting for 3.44, the law 3.11 reads:

$$\dot{m} = \dot{m}, \qquad \dot{p} = m\,(g - 2\,\Omega \times v) + \dot{m}\,(v + w) .$$

The former equality is satisfied for any value of the mass change rate $\dot{m}$. Accounting for $p = mv$, the latter one provides the equation of motion:

$$m\dot{v} = m\,(g - 2\,\Omega \times v) + \dot{m}\,w ,$$

that must be completed by a phenomenological law governing the mass change rate, depending on the kind of thrust.

3.6 Comments for experts

[Comment 1] Conversely, the changes of coordinate systems of the space-time $X' \mapsto X$ such that

$$P = \frac{\partial X}{\partial X'}$$

is a linear Galilean transformation, are the rigid motions 3.27, as it will be proved further using Frobenius method. The compatibility conditions of the system are $R = R(t)$ and 3.29.

[Comment 2] This 'space-time windows' can be seen as an intuitive interpretation of the tangent space to the space-time manifold.

[Comment 3] The gravitation Γ is in fact a symmetric connexion on the space-time manifold.

[Comment 4] This equation of motion was introduced in this form by Souriau (Formula (12.47) together with (12.44), page 133, [31], English translation [33]) from symplectic mechanics arguments.

[Comment 5] A reason to put $\Omega = 0$ is that in Schwarzschild's solution of Einstein's equations of the General Relativity, the Christoffel's symbols corresponding to Ω vanish (even if the velocity of the light is finite).

[Comment 6] It is only an hypothesis. Rather than measuring Earth's rotation frequency, Foucault's pendulum experiment allows directly measuring the Ω component of the Newtonian gravitation, as in the same way observing the free falling particles allows to measure the g component.

Chapter 4

Statics of arches, cables and beams

4.1 Statics of arches

4.1.1 Modelling of slender bodies

The aim of this chapter is to study the static equilibrium of slender 3D bodies generally speaking called **arches**, idealized by 1D material bodies from a geometrical and mechanical points of view. First of all, we model the geometry of a slender body by performing the following steps (figure 4.1):

- define a **mean line** given by the piecewise smooth map $s \mapsto \boldsymbol{Q}(s)$ where s is the arclength with respect to a given reference point of the line (there is an arbitrary in this choice which is part of the modelling),
- to each point $\boldsymbol{Q}$ of the mean line, assign a **cross-section** $\mathcal{S}_Q$ locally orthogonal to the mean line.

$r(s)$ being the position in a Galilean coordinate system of a regular point of the mean line $\boldsymbol{Q}(s)$, one can defined the tangent unit vector $\vec{\mathbf{U}}(s)$ represented in the basis of the considered coordinate system by the column:

$$U(s) = \frac{dr}{ds}(s).$$

An arch is slender in the sense that the dimensions of the cross-sections are small with respect to the one of the line. If it is seen from a long way off, it can be considered in first approximation as geometrically reduced to its mean line. In this spirit, we hope to model the internal and external efforts by an idealized sketch.

We cut through the arch along the cross-section $\mathcal{S}_Q$ and we draw the free body diagram of the upstream part $\mathcal{B}$ of the arch when running along the mean line with increasing s (figure 4.2). Let us consider the resultant torsor of the internal forces

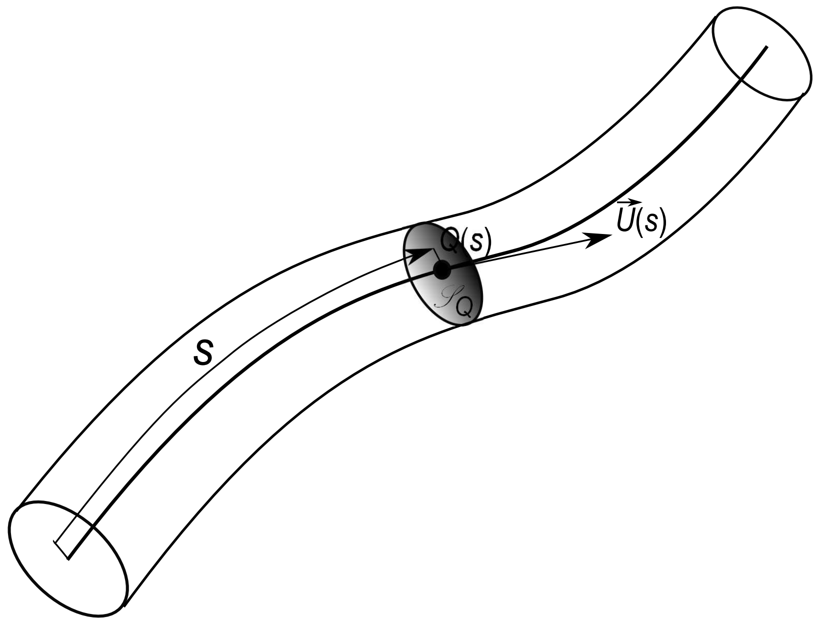

Figure 4.1: Geometric model of the arch

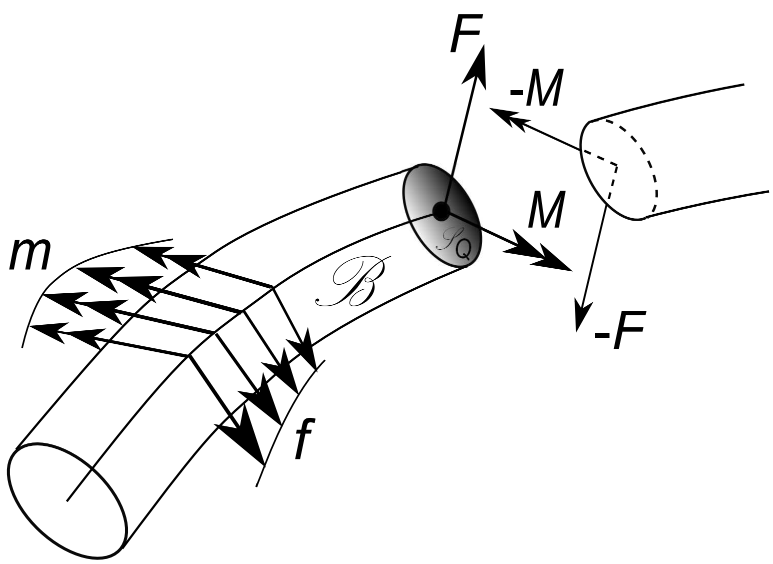

Figure 4.2: Free body diagram of the part $\mathcal{B}$ of the arch

acting through the cross-section $\mathcal{S}_Q$ upon the part $\mathcal{B}$:

$$\breve{\mu}^{int} = \begin{pmatrix} 0 & F^T \\ -F & -j(M) \end{pmatrix} ,$$

of which the force F and the moment M with respect to $\boldsymbol{Q}$ taken as origin are represented on the figure respectively by a vector and a double vector, according to an usual convention in mechanics. Owing to Newton's third law 2.6, the resultant torsor of the internal forces acting through the cross-section upon the downstream part is equal and opposite to $\breve{\mu}^{int}$ as represented on the figure. The force F is decomposed into:

- the **normal force** $N = F \cdot U$, positive in tension and negative in compression,
- and the **shear force** $T = F - (F \cdot U)U$, tangent to the cross-section,

while the moment M is decomposed into:

- the **torque** $M_t = M \cdot U$, responsable of the torsion of the arch around its mean line,
- and the **bending moment** $M_f = M - (M \cdot U)U$, which tends to modify the curvature of the mean line and to rotate the cross-section.

In the same spirit, we consider the external efforts can be modelized by a piecewise smooth distribution of exterior forces and moments of which the torsor by length unit is:

$$\frac{d\breve{\mu}^{ext}}{ds} = \begin{pmatrix} 0 & f^T \\ -f & -j(m) \end{pmatrix} ,$$

Thus the modelling defines piecewise smooth assignements $s \mapsto \breve{\mu}^{int}(s)$ and $s \mapsto \frac{d\breve{\mu}^{ext}}{ds}(s)$.

4.1.2 Local equilibrium equations of arches

To know if an arch is in equilibrium, we should verify that the resultant torsor of each of its parts is null, that is a difficult task because the number of its parts is infinite. To avoid the pitfall, we would like to establish local equations easy to test. The key idea of the classical modelling is to consider an arch slice $d\mathcal{B}$ of infinitesimal length ds, leading to differential equations. Let $\boldsymbol{Q}$ be the point on the mean line corresponding to the downstream extremity cross-section and $\boldsymbol{Q}'$ the point corresponding to the upstream extremity one that will be taken as reference origin when adding the moments.

To write the equilibrium equation of the slide, we perform the following steps:

- we draw the free body diagram of the slide (figure 4.3)
    - with the external efforts $-F$ and $-M$ acting upon the slice through the downstream extremity cross-section (according to Newton's third law 2.6),

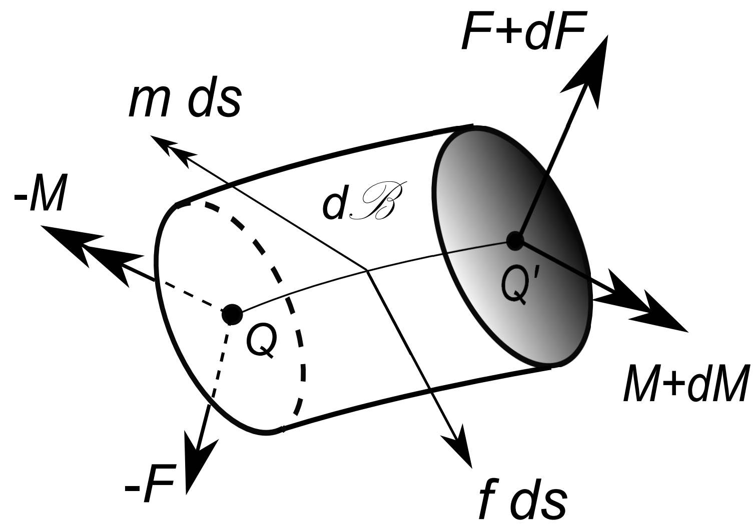

Figure 4.3: Free body diagram of the slide $d\mathcal{B}$

- ∗ the external efforts $F+dF$ and $M+dM$ acting upon the slice through the upstream extremity cross-section (taking into account their infinitesimal variation when running from $\boldsymbol{Q}$ to $\boldsymbol{Q}'$)
- ∗ and the resultants $f\,ds$ and $m\,ds$ of the external efforts distributed on a length ds with respect to the slide barycenter.

- We construct the resultant torsor of the slide. According to Subsection 2.3.1, the torsors of the forces must be given with respect to same point taken as origin, let say $\boldsymbol{Q}'$, before to calculate their sum. The position of $\boldsymbol{Q}$ with respect to itself taken as origin being $r = 0$, its position with respect to the reference origin $\boldsymbol{Q}'$ is $r' = -dr$. According to the transformation law of the torsor 2.10 for the infinitesimal spatial translation $dk = r - r' = dr$, the torsor of the external efforts $-F$ and $-M$ acting upon the slice through the downstream extremity cross-section with respect to the new position $\boldsymbol{Q}'$ is:

$$F' = -F, \qquad M' = -M + (-F) \times dr = -M + dr \times F \ ,$$

- The balance equation (2.12) of the slide reads:

$$F' + (F + dF) + f\,ds = dF + f\,ds = 0 \ ,$$

$$M' + (M + dM) + m\,ds = dr \times F + dM + m\,ds = 0 \ .$$

Dividing by ds leads to the **local equilibrium equations of arches**:

$$\boxed{\frac{dF}{ds} + f = 0} \tag{4.1}$$

$$\boxed{\frac{dM}{ds} + U \times F + m = 0} \tag{4.2}$$

It is worth to notice that the transport of the moment of the external forces from the barycenter to r' is not necessary because it generates in the moment equilibrium equation an additional infinitesimal term of the second order which can be neglected.

In the spirit of Chapter 3, we hope now to find again these equations by introducing the **covariant differential** of the torsor of internal efforts defined as:

$$\mathbf{d}\breve{\mu} = d(\breve{P}\,\breve{\mu}'\breve{P}^T)|_{r'=r} = (\breve{P}\,d\breve{\mu}'\breve{P}^T + d\breve{P}\,\breve{\mu}'\breve{P}^T + \breve{P}\,\breve{\mu}'d\breve{P}^T)\,|_{r'=r} \ .$$

When r' approaches r, $\breve{\mu}'$ approaches $\breve{\mu}$ and $\breve{P}$ approaches the identity:

$$\mathbf{d}\breve{\mu} = d\breve{\mu} + d\breve{P}\,\breve{\mu} + \breve{\mu}\,d\breve{P}^T \ . \tag{4.3}$$

Considering as previously an infinitesimal translation $dk = dr$,

$$d\breve{P} = d\begin{pmatrix} 1 & 0 \\ k & R \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ dr & 0 \end{pmatrix} \ ,$$

and owing to 12.9, the covariant differential of the torsor of the internal efforts reads:

$$\mathbf{d}\breve{\mu}^{int} = \begin{pmatrix} 0 & \mathbf{d}F^T \\ -\mathbf{d}F & -j(\mathbf{d}M) \end{pmatrix} \ , \tag{4.4}$$

with:

$$\mathbf{d}F = dF, \qquad \mathbf{d}M = dM + dr \times F \ .$$

It is easy to verify that the local equilibrium equations 4.1 and 4.2 can be recast in the following compact form:

$$\frac{\mathbf{d}\breve{\mu}^{int}}{ds} + \frac{d\breve{\mu}^{ext}}{ds} = 0 \ . \tag{4.5}$$

It is now possible to determine the distribution of internal efforts with respect to the external ones and the reactions at the ends of any part $\boldsymbol{AB}$ of the arch. Integrating 4.1 and owing to $F(s_A) = -R_A$, gives the internal force distribution:

$$F(s) = -\int_{s_A}^{s} f(s')\,ds' - R_A \ . \tag{4.6}$$

According to the previous result and owing to $M(s_A) = -M_A$, the integration of 4.2 provides the distribution of the internal moment with respect to $\boldsymbol{A}$ taken as origin:

$$M(s) = -\int_{s_A}^{s}\left(m(s') + U(s') \times \int_{s_A}^{s'} f(s'')\,ds''\right) ds' - ((r(s) - r(s_A)) \times R_A + M_A) \ . \tag{4.7}$$

Taking into account the conditions $F(s_B) = R_B$ and $M(s_B) = M_B$, the global equilibrium of the part $\boldsymbol{AB}$ reads:

$$\int_{s_A}^{s_B} f(s')\, ds' + R_A + R_B = 0 \ .$$

$$\int_{s_A}^{s_B} \left(m(s') + U(s') \times \int_{s_A}^{s_B} f(s'')\, ds'' \right) ds' + (r(s_B) - r(s_A)) \times R_A + M_A + M_B = 0 \ .$$

4.1.3 Corotational equilibrium equations of arches

In the previous section, we considered a moving coordinate system by translation of the origin along the mean line without rotation. We may also change the point of view by considering both translation and rotation. In other words, we assign to each point $\boldsymbol{Q}$ of the mean line a coordinate system r of which the origin is $\boldsymbol{Q}$, the corresponding coordinate system r' assigned to the upstream neighbour point $\boldsymbol{Q'}$ on the mean line being obtained from r by an infinitesimal Euclidean transformation $d\breve{P}$. According to 3.24, we have:

$$d\breve{P} = d \begin{pmatrix} 1 & 0 \\ k & R \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ dr & j(d\psi) \end{pmatrix} \ .$$

Calculating the covariant differential of the torsor of the internal efforts by 4.3 gives:

$$\mathbf{d}\breve{\mu}^{int} = \begin{pmatrix} 0 & (dF + j(d\psi)F)^T \\ -(dF + j(d\psi)F) & -j(dM) + dr\, F^T - F\, dr^T - j(d\psi)j(M) + j(M)j(d\psi) \end{pmatrix} \ .$$

Introducing the column:

$$\Omega = \frac{d\psi}{ds} \ , \tag{4.8}$$

Taking into account 12.9, 12.12, the linearity of j, 4.4 and 4.5, the **corotational equilibrium equations of arches** reads:

$$\boxed{\frac{dF}{ds} + \Omega \times F + f = 0} \tag{4.9}$$

$$\boxed{\frac{dM}{ds} + \Omega \times M + U \times F + m = 0} \tag{4.10}$$

It is worth to notice these equations are true for any choice of Galilean coordinate system r moving along the curve and the associate moving orthonormal basis S. The assignement $s \mapsto S(s)$ is called a **moving frame**.

4.1.4 Equilibrium equations of arches in Fresnet's moving frame

Now, we specialize the corotational equilibrium equations to a particular moving frame due to Fresnet. We construct the moving orthonormal basis by adjoining to $\vec{\mathbf{U}}$ two vectors, $\vec{\mathbf{V}}$ and $\vec{\mathbf{W}}$, respectively represented in the reference basis by the columns V and W. Differentiating $\| U \|^2 = U \cdot U = 1$ leads to:

$$U \cdot \frac{dU}{ds} = 0 \ .$$

The normed vector V in the direction of $dU \,/\, ds$, called the **normal**, is orthogonal to U. The **curvature** κ of the mean line is defined by:

$$\frac{dU}{ds} = \kappa V \ . \tag{4.11}$$

Next, we define the **binormal**:

$$W = U \times V \ , \tag{4.12}$$

orthogonal to U and V. As V and U are orthogonal, owing to 12.16, W is a unit vector. The orthonormal basis $(\vec{\mathbf{U}}, \vec{\mathbf{V}}, \vec{\mathbf{W}})$ is called **Fresnet's basis**. Differentiating $\| W \|^2 = 1$ shows that $dW \,/\, ds$ is orthogonal to W. On the other hand, differentiating 4.12 and taking into account 4.11 show that $dW \,/\, ds$ is also orthogonal to U. The **torsion** θ of the mean line is define by:

$$\frac{dW}{ds} = \theta V \ . \tag{4.13}$$

As the basis (U, V, W) is orthonormal, its variation when running along the mean line of a length ds is an infinitesimal rotation. Owing to 4.11 and 4.13, it reads:

$$(dU, dV, dW) = (U, V, W) \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \theta \\ 0 & -\theta & 0 \end{pmatrix} ds \ .$$

Then, 4.8 is:

$$\Omega = \begin{pmatrix} -\theta \\ 0 \\ \kappa \end{pmatrix} \ . \tag{4.14}$$

In Fresnet's basis, the force acting onto the cross-section is represented by the columns:

$$F = \begin{pmatrix} N \\ T_n \\ T_b \end{pmatrix} \ ,$$

where N is the normal force and T_n, T_b are respectively the shear forces with respect to the normal and the binormal, while the moment is represented by:

$$M = \begin{pmatrix} M_t \\ M_n \\ M_b \end{pmatrix} \ ,$$

where M_t is the torque and M_n, M_b are respectively the bending moments with respect to the normal and the binormal. Taking into account 4.14, the corotational equilibrium equations 4.9 and 4.10 in Fresnet's moving frame read:

$$\frac{dN}{ds} + \kappa T_n + f_t = 0 \ , \frac{dT_n}{ds} - \kappa N - \theta T_b + f_n = 0 \ , \frac{dT_b}{ds} + \theta T_n + f_b = 0 \ , \qquad (4.15)$$

$$\frac{dM_t}{ds} + \kappa M_n + m_t = 0 \ , \frac{dM_n}{ds} - \kappa M_t - \theta M_b - T_b + m_n = 0 \ , \frac{dM_b}{ds} + \theta M_n + T_n + m_b = 0 \ . \qquad (4.16)$$

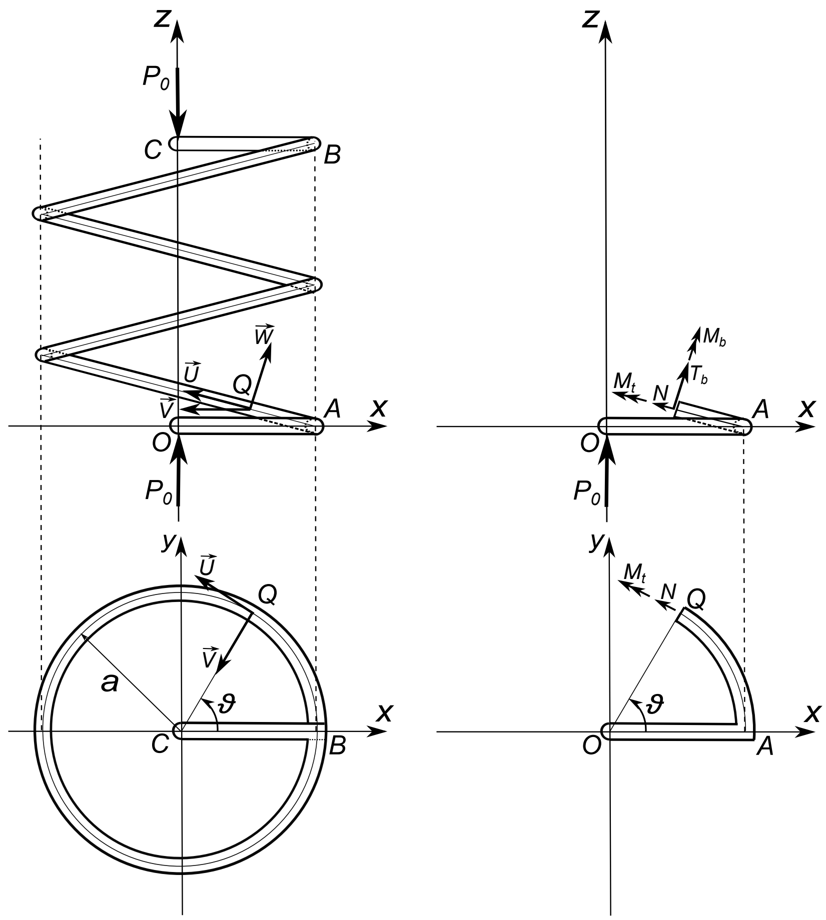

Figure 4.4: Helical coil spring

As application, let us consider a **helical coil spring** (Figure 4.4). The mean line is an helix of radius a and pitch $2\pi h$, parameterized in a reference coordinate system $\boldsymbol{O}x'y'z'$ by:

$$r'(s) = \begin{pmatrix} a\cos\vartheta \\ a\sin\vartheta \\ h\vartheta \end{pmatrix} \ .$$

where $\vartheta = s/b$ with $b = \sqrt{a^2 + h^2}$. Its curvature and torsion are:

$$\kappa = \frac{a}{b^2}, \qquad \theta = -\frac{h}{b^2} \ ,$$

and the representation of Fresnet's basis in the reference basis is given by the rotation:

$$R = (U, V, W) = \begin{pmatrix} -\frac{a}{b}\sin\vartheta & -\cos\vartheta & \frac{h}{b}\sin\vartheta \\ \frac{a}{b}\cos\vartheta & -\sin\vartheta & -\frac{h}{b}\cos\vartheta \\ \frac{h}{b} & 0 & \frac{a}{b} \end{pmatrix} . \tag{4.17}$$

The spring is compressed through horizontal arms $\boldsymbol{OA}$ and $\boldsymbol{BC}$ by two opposite vertical forces R_O and R_C of same intensity P_0 acting respectively at the extremities $\boldsymbol{O}$ and $\boldsymbol{C}$. Cutting through the arch allong the cross-section $\mathcal{S}_Q$, we draw the free body diagram of the downstream part $\boldsymbol{OAQ}$ of the body (Figure 4.5). The global force equilibrium equation 4.6 of this part gives in the reference basis the internal force F acting through the cross-section $\mathcal{S}_Q$:

$$F = -R_O = -\begin{pmatrix} 0 \\ 0 \\ P_0 \end{pmatrix} .$$

The shear force T_n is obviously null because the external force applied at $\boldsymbol{O}$ is vertical and the normal $\vec{\mathbf{V}}$ is horizontal. It is not represented on the figure. Using the rotation matrix 4.17, we obtain the internal force $\bar{F} = R^T F$ in Fresnet's basis:

$$\bar{F} = \begin{pmatrix} N \\ T_n \\ T_b \end{pmatrix} = -\frac{P_0}{b}\begin{pmatrix} h \\ 0 \\ a \end{pmatrix} . \tag{4.18}$$

The global moment equilibrium equation 4.7 of $\boldsymbol{OAQ}$ gives in the reference basis the internal moment M acting through the cross-section $\mathcal{S}_Q$ with respect to point $\boldsymbol{Q}$:

$$M = r \times R_O = aP_0 \begin{pmatrix} \sin\vartheta \\ -\cos\vartheta \\ 0 \end{pmatrix} ,$$

from which we deduce the internal moment $\bar{M} = R^T M$ in Fresnet's basis:

$$\bar{M} = \begin{pmatrix} M_t \\ M_n \\ M_b \end{pmatrix} = \frac{aP_0}{b}\begin{pmatrix} -a \\ 0 \\ h \end{pmatrix} . \tag{4.19}$$

It can be easily verified that the internal efforts 4.18 and 4.19 satisfy the corotational equilibrium equations 4.15 and 4.16.

4.2 Statics of cables

Definition 4.1 *A* ***cable*** *is an arch so thin that it cannot resist moments:*

$$M = m = 0 \ .$$

Cable is a generic word for bodies such as **threads** and **strings** which exhibit such a behaviour. The moment equilibrium equation 4.2 reads:

$$U \times F = 0 \ ,$$

from which it results the internal force F is proportional to the unit tangent vector U:

$$F = N\,U \ .$$

A cable is in equilibrium only in tension ($N > 0$). Its value and the shape of the cable are determined by solving the remaining equilibrium equation 4.1:

$$\frac{d}{ds}(N\,U) + f = 0, \ . \tag{4.20}$$

It is worth to remark that, if there is no distributed external force f on a part of a cable, $N\,U$ is constant. Hence this part is straight and the tension N is constant, in agreement with the experience according to which a taut cable is straight.

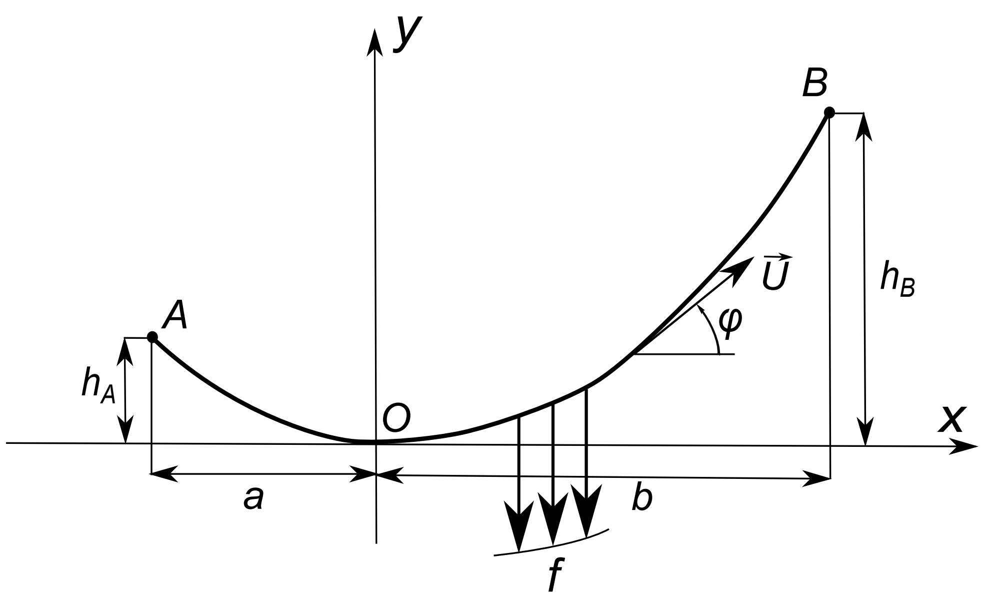

Figure 4.5: Suspension bridge

As example, let us consider a cable $\boldsymbol{AB}$ in the $\boldsymbol{Oxy}$ plane (figure 4.5). The horizontal distance between the extremities $\boldsymbol{A}$ and $\boldsymbol{B}$ is L and their heights with respect to the $\boldsymbol{O}x$ axis are respectively h_A and h_B. The lack of moment at the extremities is symbolized by hinges. Under a vertical load f of intensity q by length unit and opposite direction to $\boldsymbol{O}y$, the cable is supposed to be tangent to the horizontal axis at the origin. φ being the angle of $\vec{\boldsymbol{U}}$ with respect to the horizontal axis, the force equilibrium reads:

$$\frac{d}{ds}(N \cos\varphi) = 0 \ , \tag{4.21}$$

$$\frac{d}{ds}(N \sin\varphi) = q \ . \tag{4.22}$$

As application, let us consider a suspension cable of a **suspension bridge**. Introducing p such that:

$$p(x)\, dx = q(s)\, ds \ ,$$

the last equation reads:

$$\frac{d}{dx}(N \sin\varphi) = p \ . \tag{4.23}$$

integrating equations 4.21 and 4.23 gives:

$$N \cos\varphi = C_0, \qquad N \sin\varphi = \int_0^x p(x')\, dx' + C_1 \ ,$$

where C_0 and C_1 are constant. Elimilation N between these equations leads to:

$$\frac{dy}{dx} = \tan\varphi = \frac{1}{C_0}\,(\int_0^x p(x')\, dx' + C_1) \ ,$$

and by a new integration, to the shape of the cable:

$$y(x) = \frac{1}{C_0}\,(\int_0^x dx'' \int_0^{x''} p(x')\, dx' + C_1 x) + C_2 \ ,$$

where C_2 is constant. The proper weight of the cable being negligeable, it is subjected only to the weight of the deck hung on vertical suspenders. This load can be modeled by vertical forces of uniform intensity p by unit of horizontal length, that gives, taking into account the conditions $y(0) = 0$ and $\frac{dy}{dx}(0) = 0$:

$$y(x) = \frac{p}{2C_0} x^2 \ .$$

The three unknowns, the distances a and b of $\boldsymbol{A}$ and $\boldsymbol{B}$ to the origin $\boldsymbol{O}$ and C_0 are determined by the additional conditions:

$$a + b = L, \qquad h_A = \frac{p}{2C_0} a^2, \qquad h_B = \frac{p}{2C_0} b^2 \ .$$

4.3 Statics of trusses and beams

4.3.1 Traction of trusses

Definition 4.2 *A **truss** is a straight arch axially loaded and ended by hinges.*

They are not subjected to moments, then the internal force F is proportional to the unit tangent vector U, as for the cables. However, unlike the last ones that are in static equilibrium only in tension, the trusses exhibit an internal force F in tension or compression. It is determined by the equilibrium equation 4.20. As application, let us consider a **drilling riser** hung to an offshore platform and loaded by its own weight of intensity p (Figure 4.6). The truss is fixed to the platform with a hinge while the other extremity is effort free.

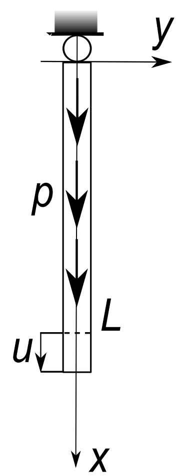

Figure 4.6: Drilling riser hung to an offshore platform

Assuming the truss is deformed along its axis, the tangent unit vector is vertical. The force equilibrium equation 4.20 is reduced to:

$$\frac{dN}{ds} + p = 0 \ .$$

Integrating it with null value at the free extremity $x = L$ gives:

$$N(s) = p(L - s) \ . \tag{4.24}$$

To determine the displacement of the points of the mean line, we need some additional assumptions concerning the behaviour of the material constituting the body. If the deformation is small, many materials such as metals are **elastic**. They obey **Hooke's law** (1678):

Law 4.3 *The effort is proportional to the corresponding deformation ("ut tensio, sic vis").*

Under a normal force N, the material undergoes a deformation measured by the **extension**:

$$\varepsilon_x = \frac{ds - dx}{dx} = \frac{ds}{dx} - 1 \ . \tag{4.25}$$

The **elasticity** law reads:

$$N = K_t \, \varepsilon_x \ , \tag{4.26}$$

where the **stiffness** K_t *a priori* depends on the material properties and the cross-section geometry. For small extensions as in elasticity, the displacements are so small that we can approximate the normal force 4.24 by:

$$N(s(x)) \cong p(L - x) \ .$$

Combining with 4.25 and 4.26 leads to the equation:

$$\frac{ds}{dx} = 1 + \frac{p}{K_t}(L - x) \ .$$

of which, owing to the condition $s(0) = 0$, the solution is for an uniform stiffness K_t:

$$s(x) = x + \frac{p}{K_t}\left(Lx - \frac{x^2}{2}\right) \ .$$

The displacement at the free extremity of the truss is:

$$u = s(L) - L = \frac{p\,L^2}{2K_t} \ .$$

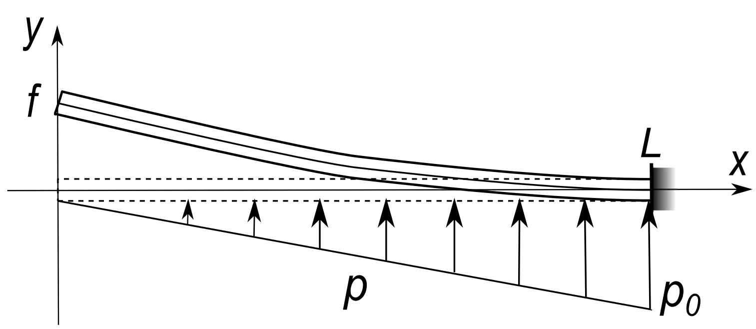

Figure 4.7: Cantilever beam

4.3.2 Bending of beams

Definition 4.4 *A* ***beam*** *is a straight arch transversally loaded.*

As application, we consider a **Cantilever beam** subjected to a linearly distributed load, built-in at the left hand end, effort free at the other one (Figure 4.7). Intuitively, the transversal load induces a vertical **deflexion** $y(x)$ in the direction of the load. For easiness, we use simplified notations for derivatives:

$$y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2y}{dx^2}, \qquad \ldots$$

The arclength element is:

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + (y')^2} \ ,$$

Fresnet's basis is represented in the reference frame by the rotation matrix:

$$R = (U, V, W) = \begin{pmatrix} \frac{1}{\sqrt{1+(y')^2}} & -\frac{y'}{\sqrt{1+(y')^2}} & 0 \\ \frac{y'}{\sqrt{1+(y')^2}} & \frac{1}{\sqrt{1+(y')^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.27}$$

The torsion is null and the curvature is:

$$\kappa = \frac{y''}{(1 + (y')^2)^{3/2}} \ .$$

using the rotation matrix 4.27, we obtain the internal force $F' = RF$ and internal moment $M' = RM$ in the reference frame:

$$T_x = \frac{1}{\sqrt{1 + (y')^2}}(N - y'T_n), \qquad T_y = \frac{1}{\sqrt{1 + (y')^2}}(y'N + T_n), \qquad T_z = T_b \ ,$$

$$M_x = \frac{1}{\sqrt{1 + (y')^2}}(M_t - y'M_n), \qquad M_y = \frac{1}{\sqrt{1 + (y')^2}}(y'M_t + M_n), \qquad M_z = M_b \ .$$

For elastic beams, the deflexions are so small that we can assume the sloop y' is negligeable with respect to the unity, leading to the following approximations:

$$ds \cong dx, \qquad \kappa \cong y'', \qquad T_x \cong N, \qquad T_y \cong T_n, \qquad M_x \cong M_t, \qquad M_y \cong M_n \ .$$

Hence, the force equilibrium equation 4.1 in the reference frame gives:

$$\frac{dN}{dx} = 0, \qquad \frac{dT_n}{dx} + p = 0, \qquad \frac{dT_b}{dx} = 0 \ . \tag{4.28}$$

Integrating with null values at the free extremity $x = L$, we conclude that N and T_b are identically null. The force equilibrium equation 4.2 in the reference frame gives:

$$\frac{dM_t}{dx} = 0, \qquad \frac{dM_n}{dx} = 0, \qquad \frac{dM_b}{dx} + T_n = 0 \ . \tag{4.29}$$

Integrating with null values at the free extremity $x = 0$, we conclude that M_t and M_n are identically null. Intuitively, under a bending moment M_b, the straight beam is curved. Applying Hooke's law 4.3, we assume that:

$$M_b = K_b \, \kappa \ , \tag{4.30}$$

where the **stiffness** K_b *a priori* depends on the material properties and the cross-section geometry. According with the above approximation on the curvature, combining this relation with the last condition in 4.29, the shear force is for an uniform stiffness K_b:

$$T_n = -K_b y''' \ ,$$

and the second condition in 4.28 gives the **elastica** equation:

$$y'''' = \frac{p}{K_b} \ .$$

Integrating it with the conditions $y'''(0) = -T_n(0)/K_b = 0$, $y''(0) = M_b(0)/K_b = 0$ at the free extremity and $y(L) = y'(L) = 0$ at the built-in support, we obtain the shape of the deformed beam:

$$y(x) = \frac{p_0}{120 K_b L}(x^5 - 5L^4 x + 4) \ .$$

The deflexion at the free extremity $x = 0$ is:

$$f = \frac{p_0 L^4}{30 K_b} \ .$$

4.4 Exercises

4.4.1 Funicular

Show the shape of a cable in equilibrium under its proper weight, vertically oriented and with uniform force intensity q by unit length is a **funicular** (or **catenary**):

$$y(x) = \frac{C_0}{q} \left(\cosh \frac{qx}{C_0} - 1 \right) \ .$$

Hint. Integrating 4.21 and 4.22, next using $ds = \sqrt{dx^2 + dy^2}$, show that:

$$x(s) = \int_0^s \frac{ds''}{\sqrt{1 + \left(\frac{1}{C_0} \left(\int_0^{s''} q(s') \, ds' + C_1 \right) \right)}} \ .$$

Chapter 5

Dynamics of rigid bodies

5.1 Kinetic co-torsor

5.1.1 Lagrangean coordinates

A rigid body is such that all material lengths and angles remain unchanged by its motion. The Galilean coordinate systems are natural tools to model the rigid body motions, especially those X' in which every point of the body of position s' at any time t' is at rest:

$$v' = \frac{ds'}{dt'} = 0 \ . \tag{5.1}$$

We say the s'^j are the **Lagrangean coordinates** or the **material coordinates** of the point. Of course, this Lagrangean or material representation is not unique. For a given body, the change of Lagrangean coordinate systems are the time independent Euclidean transformation. Although it is more or less obvious, let us give a proof. If s is the position of the points at the time t in another system X' of Lagrangean coordinates, it is related to $\bar{X}'$ by a change 3.27 composed of a rigid motion and a change clock:

$$s' = Q(t)\,\bar{s}' + s_0(t), \qquad \bar{t}' = t' + \tau \ .$$

Owing to the velocity addition formula 1.13, the corresponding velocity of transport must vanish:

$$u = v' - Q\,\bar{v}' = 0 \ .$$

Taking into account 3.29, it holds for any s and t:

$$v = u = \dot{s}_0(t) + j(\varpi(t))(s' - s_0(t)) = 0 \ ,$$

where $\dot{Q} = j(\varpi)Q$. This affine map of s' is identically null if $\dot{s}_0(t) - j(\varpi(t))s_0(t)$ and $j(\varpi(t))$ vanish, then $\varpi(t)$ and $\dot{s}_0(t)$ so are. Taking into account 3.25, Q and s_0 are time independent, that achieves the proof:

$$s' = Q\,\bar{s}' + s_0 \ . \tag{5.2}$$

5.1.2 Eulerian coordinates

In an arbitrary Galilean coordinate system X, the particle of Lagrangean coordinates s' has at time t:

- the position of which the components r^i are called **Eulerian coordinates** or **spatial coordinates**:

$$r = R(t)s' + r_0(t) \ , \tag{5.3}$$

- and, owing to 1.13 and 5.1, the velocity 3.29:

$$v = u = \dot{r}_0(t) + \varpi(t) \times (r - r_0(t)) \ . \tag{5.4}$$

where ϖ is Poisson's vector defined by 3.25:

$$\dot{R} = j(\varpi)\, R \ . \tag{5.5}$$

Definition 5.1 *A* ***vector field*** *is a distribution af vector in a given region. Vector fields can be graphically represented drawing for each point* $\boldsymbol{Q}$ *of coordinates* r *the corresponding bound vector as an arrow of origin* $\boldsymbol{Q}$*.*

For instance, we could display the velocity field v of a rigid body at each time t but this Eulerian representation is blurred by the motion of the body itself. To avert this drawback, it is convenient to pull back the velocity onto the body at rest by drawing for each point $\boldsymbol{Q'}$ of Lagrangean coordinates s' the bound vector $R^T v$. The velocity field 5.4 is determined by the velocity $\dot{r}_0$ of the origin r_0 and Poisson's vector ϖ pulled back as:

$$\dot{r}'_0 = R^T \dot{r}_0, \qquad \varpi' = R^T \varpi \ . \tag{5.6}$$

5.1.3 Co-torsor

Let us consider now any other Lagrangean coordinate system $\bar{s}'$ such that:

$$r = \bar{R}(t)\bar{s}' + \bar{r}_0(t) \ ,$$

The reader can easily deduce from 5.3 and the previous relation that s' and $\bar{s}'$ are related by 5.2 with:

$$Q = R^T \bar{R} \ , \tag{5.7}$$

$$s_0 = R^T(\bar{r}_0 - r_0) \ . \tag{5.8}$$

Firstly, let us observe that Poisson's vector is independent of the choice of the Lagrangean coordinate system because Q is time independent:

$$j(\bar{\varpi}) = \dot{\bar{R}}\bar{R}^T = (\dot{R}Q)\,(RQ)^T = \dot{R}R^T = j(\varpi)$$

hence $\bar{\varpi} = \varpi$ because the map j is regular. By pull back onto the new Lagrangean coordinate system, one has:

$$\dot{\bar{r}}'_0 = \bar{R}^T \dot{r}_0, \qquad \bar{\varpi}' = \bar{R}^T \varpi \ . \tag{5.9}$$

From the previous relation, 5.6 and 5.7, we deduce easily:

$$\bar{\varpi}' = Q^T \varpi' \ . \tag{5.10}$$

Next, taking into account that s_0 is time independent, 5.8 leads to:

$$\dot{\bar{r}}_0 = \dot{R}\, s_0 + \dot{r}_0 \ .$$

Combining with 5.9 gives:

$$\dot{\bar{r}}'_0 = \bar{R}^T(\dot{R}\, s_0 + \dot{r}_0) \ ,$$

and because of 5.7 and 5.6:

$$\dot{\bar{r}}'_0 = Q^T(\dot{r}'_0 + R^T \dot{R}\, s_0) \ ,$$

but one has, owing to 5.5 and 12.20:

$$R^T \dot{R} = R^T \dot{R} R^T R = R^T j(\varpi) R = j(R^T \varpi) \ ,$$

hence, taking into account 5.6, we obtain:

$$\dot{\bar{r}}'_0 = Q^T(\dot{r}'_0 + \varpi' \times s_0) \ . \tag{5.11}$$

As in Statics, we work temporarily with the 4-column:

$$\breve{X} = \begin{pmatrix} 1 \\ s' \end{pmatrix} \ ,$$

and the 4×4 matrix:

$$\breve{P} = \begin{pmatrix} 1 & 0 \\ s_0 & Q \end{pmatrix} \ , \tag{5.12}$$

so the Euclidean transformation 5.2 looks like a simple regular linear transformation:

$$\breve{X} = \breve{P} \breve{\bar{X}} \ . \tag{5.13}$$

Definition 5.2 *A* ***co-torsor*** $\breve{\bar{\gamma}}$ *is an object represented in a coordinate system by a skew-symmetric* 4×4 *matrix:*

$$\boxed{\breve{\gamma} = \begin{pmatrix} 0 & v^T \\ -v & -j(\omega) \end{pmatrix} \ ,} \tag{5.14}$$

where $v \in \mathbb{R}^3$ *is its* ***velocity****,* $\omega \in \mathbb{R}^3$ *is its* ***spin****, and of which the components, under the Euclidean transformation 5.13, are modified according to the transformation law:*

$$\breve{\gamma} = \breve{P}^{-T} \breve{\bar{\gamma}} \breve{P}^{-1} \ . \tag{5.15}$$

Applying the rules of the matrix calculus, it is worth to notice that if $\breve{\bar{\gamma}}$ is skew-symmetric, $\breve{\gamma}$ given by 5.15 so is. Under an Euclidean transformation (and more generally under an affine transformation), the skew-symmetry property is preserved, that ensures the consistency of the definition. By inversion of 5.15, one has:

$$\breve{\bar{\gamma}} = \breve{P}^T \breve{\gamma} \breve{P} \ , \tag{5.16}$$

Taking into account 5.12 and 5.14, the reader can easily verify the transformation law of the co-torsor components:

$$\bar{v} = Q^T (v + \omega \times s_0), \qquad \bar{\omega} = Q^T \omega \ , \tag{5.17}$$

It is also easy to find two invariants under Euclidean transformations:

- the norm of the spin: $\parallel \omega \parallel$,
- the dot product of the velocity and the moment: $v \cdot \omega$.

The linear space $\mathbb{M}_{44}^{skew}$ of the 4×4 skew-symmetric matrices is of dimension 6. Let $\mathbf{T}_s^*$ be the set of co-torsors $\boldsymbol{\gamma}$, in one-to-one correspondance with the skew-symmetric 4×4 matrices 5.14. Thanks to this map, $\mathbf{T}_s^*$ is a linear space of dimension 6 if we define by structure transport the addition of co-torsors and the multiplication of a co-torsor by a scalar.

It is worth to observe the similarity and discrepancy between co-torsors and torsors. Formaly comparing the transformation laws of their respective components, namely 2.10 and 5.17, we observe that the velocity v is the analogous of the moment M and the spin ϖ of the force F but, comparing the positions of v and ϖ in $\breve{\gamma}$ to the ones of M and F in $\breve{\mu}$, the analogy is no more relevant. In fact, the co-torsors ar objects distinct from the torsors [Comment 1].

On this ground, the previous study shows that the velocity field of a given rigid body can be characterized in the Lagrangean coordinates s' by a co-torsor of velocity $\dot{r}'_0$ and spin ϖ':

$$\breve{\gamma} = \begin{pmatrix} 0 & \dot{r}_0'^T \\ -\dot{r}'_0 & -j(\varpi') \end{pmatrix} \ , \tag{5.18}$$

because their transformation laws 5.11 and 5.10 are just the ones 5.17 of a co-torsor. We call it the **kinetic co-torsor of a rigid body** because –r_0 being fixed– it contains the essential information to know, through 5.4, the velocity field of the body in its motion around it.

5.2 Dynamical torsor

5.2.1 Total mass and mass-centre

According to the experimental observations, we claim that:

Law 5.3 *The mass is an extensive quantity.*

On this ground, we define the dynamical torsor of a body occupying a continuous domain $\mathcal{B}$ in a given Lagrangean coordinate system. The **total mass** is:

$$m_{\mathcal{B}} = \iiint_{\mathcal{B}} dm(s') \ .$$

As noticed at Section 3.1.2, the elementary particles as an electron have *a priori* a spin –even in classical mechanics– but, as far as we are concerned here, it will be neglected. According to 3.15, the elementary dynamical torsor of an elementary volume $d\mathcal{B}(s')$ around s' of infinitesimal mass $dm(s')$ has the reduced form in the Lagrangean coordinate system (because $d\mathcal{B}(s')$ is at rest in it):

$$d\tilde{\mu}'(s') = \begin{pmatrix} 0 & dm(s') & 0 \\ -dm(s') & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \qquad (5.19)$$

Let us consider now the point of coordinates:

$$s'_{\mathcal{B}} = \frac{1}{m_{\mathcal{B}}} \iiint_{\mathcal{B}} s' \, dm(s') \ .$$

Owing to the linearity of the integral and 5.2, it is represented in another Lagrangean coordinate system by:

$$\bar{s}'_{\mathcal{B}} = Q^T(s'_{\mathcal{B}} - s_0) \ ,$$

hence, its coordinates change as the components of a point called the **mass-centre** of $\mathcal{B}$. Its definition does not depend of the choice of the Lagrangean coordinate system. The Lagrangean coordinate systems X' of which the space origin is the mass-centre are called **barycentric coordinate systems** and, of course, one has in such coordinate systems:

$$s'_{\mathcal{B}} = \frac{1}{m_{\mathcal{B}}} \iiint_{\mathcal{B}} s' \, dm(s') = 0 \ . \qquad (5.20)$$

The changes of barycentric coordinate systems are time independent rotations.

5.2.2 The rigid body as a particle

According to Newton's third law 2.6, the resultant torsor is an extensive quantity. On this ground, we claim that:

Law 5.4 *The dynamical torsor is an extensive quantity.*

This law must be applied with caution. Inded, we wish calculating the resultant dynamical torsor of the rigid body. In order to be summed, the elementary dynamical torsors of the elementary volumes must be represented in a common coordinate system X (in the same spirit as in Section 2.3.1). For this aim, we apply the boost method of Section 3.1.2 to the reduced form 5.19, given in a particular coordinate system X' for which the elementary volume $d\mathcal{B}(s')$, assimilated to a particle, is at rest at the position $r' = 0$. Let us consider the other coordinate system $X = PX' + C$ with the Galilean boost:

$$v = \frac{dr}{dt} \ , \tag{5.21}$$

and a translation k of the origin $r' = 0$ at $r = k$ (hence $\tau = 0$ and $R = 1_{\mathbb{R}^3}$). Performing the matrix product 3.2 applied to 5.19 gives the new components of the elementary dynamical torsor:

$$d\tilde{\mu}(s') = \begin{pmatrix} 0 & dm(s') & (dp(s'))^T \\ -dm(s') & 0 & -(dq(s'))^T \\ -dp(s') & dq(s') & -j(dl(s')) \end{pmatrix} . \tag{5.22}$$

where:

$$dp(s') = v\, dm(s'), \qquad dq(s') = r\, dm(s'), \qquad dl(s') = r \times v\, dm(s') \ .$$

We are now allowed to calculate the sum of these elementary contributions because they are given with respect to a common coordinate system X of arbitrary origin. This leads to the following definition:

Definition 5.5 *The* ***dynamical torsor of a body*** $\mathcal{B}$ *is the integral of the elementary dynamical torsors of every infinitesimal parts:*

$$\tilde{\mu}(\mathcal{B}) = \iiint_{\mathcal{B}} d\tilde{\mu}(s') \ = \begin{pmatrix} 0 & m_{\mathcal{B}} & p_{\mathcal{B}}^T \\ -m_{\mathcal{B}} & 0 & -q_{\mathcal{B}}^T \\ -p_{\mathcal{B}} & q_{\mathcal{B}} & -j(l_{\mathcal{B}}) \end{pmatrix} \ ,$$

where:

$$p_{\mathcal{B}} = \iiint_{\mathcal{B}} v\, dm(s'), \qquad q_{\mathcal{B}} = \iiint_{\mathcal{B}} r\, dm(s'), \qquad l_{\mathcal{B}} = \iiint_{\mathcal{B}} r \times v\, dm(s')$$

In the arbitrary Galilean coordinate system X, the particle of barycentric coordinates s' has at time t the position:

$$r = R(t)s' + r_{\mathcal{B}}(t) \ , \tag{5.24}$$

and the velocity 5.21. Using the mass-centre of Eulerian coordinates $r^i_{\mathcal{B}}$, it is possible to decompose the dynamic of a rigid body into the overall motion of the body with the mass concentrated at its mass-centre and the motion of the body around it. This is the purpose of **König's first theorem**:

Theorem 5.6 *The motion of a rigid body is equivalent to the one of a particle of mass $m_{\mathcal{B}}$, position $r_{\mathcal{B}}$, velocity $\dot{r}_{\mathcal{B}}$ and a spin angular momentum $l_{0\mathcal{B}}$ linearly depending on Poisson's vector ϖ, the components of its dynamical torsor being:*

- *the mass:* $m_{\mathcal{B}}$*,*
- *the linear momentum:* $p_{\mathcal{B}} = m_{\mathcal{B}}\dot{r}_{\mathcal{B}}$*,*
- *the passage:* $q_{\mathcal{B}} = m_{\mathcal{B}} r_{\mathcal{B}}$*,*
- *the angular momentum:* $l_{\mathcal{B}} = r_{\mathcal{B}} \times m_{\mathcal{B}}\dot{r}_{\mathcal{B}} + \mathcal{J}_{\mathcal{B}}\varpi$*,*

when working in barycentric coordinates.

Proof. Owing to the velocity addition formula 1.13 and the formula 3.29 for the velocity of transport, the velocity in the considered Eulerian coordinate system X is:

$$v = u + R\,v' = u = \dot{r}_{\mathcal{B}}(t) + \varpi(t) \times (r - r_{\mathcal{B}}(t)) \ .$$

Combining with 5.24 gives:

$$v = \dot{r}_{\mathcal{B}}(t) + \varpi(t) \times (R(t)s') \ . \tag{5.25}$$

Introducing this expression into the one of the linear momentum $p_{\mathcal{B}}$, one has:

$$p_{\mathcal{B}} = \dot{r}_{\mathcal{B}}(t) \iiint_{\mathcal{B}} dm(s') + \varpi(t) \times \left(R(t) \iiint_{\mathcal{B}} s'\, dm(s') \right) \ .$$

Because we are working in barycentric coordinates, 5.20 leads to:

$$p_{\mathcal{B}} = m_{\mathcal{B}}\dot{r}_{\mathcal{B}} \ . \tag{5.26}$$

Introducing expression 5.24 into the one of the passage $q_{\mathcal{B}}$, one has:

$$q_{\mathcal{B}} = r_{\mathcal{B}}(t) \iiint_{\mathcal{B}} dm(s') + R(t) \iiint_{\mathcal{B}} s'\, dm(s') \ .$$

Owing to 5.20 leads to:

$$q_{\mathcal{B}} = m_{\mathcal{B}} r_{\mathcal{B}} \ . \tag{5.27}$$

Once again introducing expressions 5.24 and 5.25 into the one of the angular momentum $l_{\mathcal{B}}$, one has:

$$\begin{aligned} l_{\mathcal{B}} &= r_{\mathcal{B}} \times \dot{r}_{\mathcal{B}}(t) \iiint_{\mathcal{B}} dm(s') + r_{\mathcal{B}} \times \left(\varpi(t) \times \left(R(t) \iiint_{\mathcal{B}} s'\, dm(s') \right) \right) \\ &+ \left(R(t) \iiint_{\mathcal{B}} s'\, dm(s') \right) \times \dot{r}_{\mathcal{B}}(t) \\ &+ \iiint_{\mathcal{B}} (R(t)s') \times (\varpi(t) \times (R(t)s'))\, dm(s') \ . \end{aligned} \tag{5.28}$$

Owing to 5.20, the second and third terms of the left hand side vanish and:

$$l_{\mathcal{B}} = l_{0\mathcal{B}} + r_{\mathcal{B}} \times m_{\mathcal{B}}\dot{r}_{\mathcal{B}} \ ,$$

where the second term of the right hand side is the orbital angular momentum and the spin angular momentum:

$$l_{0\mathcal{B}} = \iiint_{\mathcal{B}} (R(t)s') \times (\varpi(t) \times (R(t)s'))\, dm(s') \ . \tag{5.29}$$

is a linear function of Poisson's vector $\varpi(t)$:

$$l_{0\mathcal{B}} = \mathcal{J}_{\mathcal{B}}(t)\, \varpi(t) \ , \tag{5.30}$$

that achieves the proof. ■

In short, the dynamical torsor of a body depends on:

- the whole motion of the body through the mass-centre position $r_{\mathcal{B}}$,
- the motion of the body arround it through $\dot{r}_{\mathcal{B}}$ and ϖ or, equivalently, through the kinetic co-torsor which describes this motion,
- the geometrical and material characteristics of the body, the total mass and the moment of inertia matrix.

5.2.3 The moment of inertia matrix

Now we would like to calculate explicitly the 3×3 matrix $\mathcal{J}_{\mathcal{B}}(t)$.

Theorem 5.7 *The linear map from Poisson's vector ϖ onto the spin angular momentum $l_{0\mathcal{B}}$ is:*

$$\mathcal{J}_{\mathcal{B}}(t) = R(t)\mathcal{J}'_{\mathcal{B}}(R(t))^T \tag{5.31}$$

with the time-independent ***moment of inertia matrix****:*

$$\mathcal{J}'_{\mathcal{B}} = \iiint_{\mathcal{B}} (\| s' \|^2 \, 1_{\mathbb{R}^3} - s' s'^T)\, dm(s') \tag{5.32}$$

Proof. Using 5.6 and 12.19, we have:

$$(R\, s') \times (\varpi \times (R\, s')) = (R\, s') \times ((R\, \varpi') \times (R\, s')) = R\, (s' \times (\varpi' \times s')) \ .$$

Taking into account the vector triple product 12.14 and 5.6, one has:

$$\begin{aligned}(R\, s') \times (\varpi \times (R\, s')) &= R\, (\| s' \|^2 \, \varpi' - (s' \cdot \varpi') s') \\ &= R(t)\, (\| s' \|^2 \, 1_{\mathbb{R}^3} - s' s'^T)\, (R(t))^T \varpi(t) \ .\end{aligned}$$

Introducing this expression into 5.29 and identifying with 5.30, we obtain 5.31 with 5.32, that achieves the proof. ■

Denoting x', y', z' the Lagrangean coordinates, the moment of inertia matrix is:

$$\mathcal{J}'_{\mathcal{B}} = \begin{pmatrix} A' & -H' & -G' \\ -H' & B' & -F' \\ -G' & -F' & C' \end{pmatrix} ,$$

where the moments of inertia with respect to the coordinate axes are:

$$A' = \iiint_{\mathcal{B}} (y'^2 + z'^2)\, dm, \qquad B' = \iiint_{\mathcal{B}} (z'^2 + x'^2)\, dm, \qquad C' = \iiint_{\mathcal{B}} (x'^2 + y'^2)\, dm ,$$

and the product of inertia are:

$$F' = \iiint_{\mathcal{B}} y'z'\, dm, \qquad G' = \iiint_{\mathcal{B}} z'x'\, dm, \qquad H' = \iiint_{\mathcal{B}} x'y'\, dm .$$

The moments of inertia matrix $\mathcal{J}'_{\mathcal{B}}$ is symmetric then diagonalizable with real eigenvalues and the corresponding matrix $P = (V_1, V_2, V_3)$ is orthogonal. The eigenvectors V_1, V_2, V_3 are mutually orthogonal and of unit norm. The corresponding axis are called **principal axis of inertia**. The eigenvalues are the **principal moments of inertia** A, B, C.

Using 5.24, it is also worth to notice that the moment of inertia matrix 5.31 can be recast as:

$$\mathcal{J}_{\mathcal{B}} = \iiint_{B} (\| r - r_{\mathcal{B}} \|^2\, 1_{\mathbb{R}^3} - (r - r_{\mathcal{B}})(r - r_{\mathcal{B}})^T)\, dm(r) , \tag{5.33}$$

where B is the image of $\mathcal{B}$ through the map $s' \longmapsto r$. Then we define the matrix of inertia at the origin $r = 0$ as:

$$\mathcal{J}_{\mathcal{B}O} = \iiint_{B} (\| r \|^2\, 1_{\mathbb{R}^3} - r r^T)\, dm(r) , \tag{5.34}$$

and we prove **Huygens' theorem**:

Theorem 5.8 *The matrix of inertia $\mathcal{J}_{\mathcal{B}}$ is transported at the origin according to:*

$$\mathcal{J}_{\mathcal{B}O} = \mathcal{J}_{\mathcal{B}} + m_{\mathcal{B}}(\| r_{\mathcal{B}} \|^2\, 1_{\mathbb{R}^3} - r_{\mathcal{B}} r_{\mathcal{B}}^T) . \tag{5.35}$$

Proof. Expanding 5.33 and taking into account 5.34, it holds:

$$\begin{aligned} \mathcal{J}_{\mathcal{B}} &= \mathcal{J}_{\mathcal{B}O} + m_{\mathcal{B}}(\| r_{\mathcal{B}} \|^2\, 1_{\mathbb{R}^3} - r_{\mathcal{B}} r_{\mathcal{B}}^T) \\ &+ \left(\iiint_{B} r\, dm(r) \right) r_{\mathcal{B}}^T + r_{\mathcal{B}} \left(\iiint_{B} r\, dm(r) \right)^T - 2 \left(\left(\iiint_{B} r\, dm(r) \right) \cdot r_{\mathcal{B}} \right) 1_{\mathbb{R}^3} \end{aligned} \tag{5.36}$$

But, owing to 5.24 and 5.20, it holds;

$$\iiint_{\mathcal{B}} r\, dm(r) = R(t) \iiint_{\mathcal{B}} s'\, dm(s') + m_{\mathcal{B}} r_{\mathcal{B}} = m_{\mathcal{B}} r_{\mathcal{B}} \ .$$

Using this last relation to symplify 5.36 leads to:

$$\mathcal{J}_{\mathcal{B}} = \mathcal{J}_{\mathcal{B}O} - m_{\mathcal{B}}(\| r_{\mathcal{B}} \|^2 \, 1_{\mathbb{R}^3} - r_{\mathcal{B}} r_{\mathcal{B}}^T) \ ,$$

that achieves the proof. ■

Considering a change of coordinate system $r = Rr'$, the reader can verify easily that the moment of inertia matrix is transformed according to a formula similar to 5.31:

$$\mathcal{J}_{\mathcal{B}O} = R\, \mathcal{J}'_{\mathcal{B}O}\, R^T \ . \tag{5.37}$$

5.2.4 Kinetic energy of a body

The kinetic energy 3.73 of the elementary volume $d\mathcal{B}(s')$ around s' is:

$$de(s') = \frac{1}{2} \| v \|^2 \ dm(s') \ .$$

Assuming –as the mass– the kinetic energy is an extensive quantity, the **total kinetic energy** is:

$$e_{\mathcal{B}} = \iiint_{\mathcal{B}} \frac{1}{2} \| v \|^2 \ dm(s') \ . \tag{5.38}$$

König's second theorem is a straighforward extention of the first theorem:

Theorem 5.9 *The kinetic energy of the body is decomposed into the energy of the body with the mass $m_{\mathcal{B}}$ concentrated at its mass-centre and the one relative to the motion of the body around it:*

$$e_{\mathcal{B}} = \frac{1}{2} m_{\mathcal{B}} \| \dot{r}_{\mathcal{B}} \|^2 + \frac{1}{2} \varpi \cdot (\mathcal{J}_{\mathcal{B}} \varpi) \ ,$$

when working in barycentric coordinates.

Proof. Taking into account expression 5.25 of the velocity in the Eulerian coordinate system, one has:

$$e_{\mathcal{B}} = \iiint_{\mathcal{B}} \frac{1}{2} \| \dot{r}_{\mathcal{B}}(t) + \varpi(t) \times (R(t)s') \|^2 \ dm(s') \ .$$

Owing to 5.20 leads to:

$$e_{\mathcal{B}} = \frac{1}{2} m_{\mathcal{B}} \| \dot{r}_{\mathcal{B}} \|^2 + \iiint_{\mathcal{B}} \frac{1}{2} \| \varpi(t) \times (R(t)s') \|^2 \ dm(s') \ . \tag{5.39}$$

Taking into account 12.16, one has:

$$\begin{aligned}
\parallel \varpi \times (R\,s') \parallel^2 &= \parallel \varpi \parallel^2 \parallel R\,s' \parallel^2 -(\varpi \cdot (R\,s')^2 \\
&= \varpi \cdot \left[\parallel R\,s' \parallel^2 \varpi - R\,s'((R\,s') \cdot \varpi)\right] \\
&= \varpi \cdot \left[\parallel s' \parallel^2 1_{\mathbb{R}^3} - R\,s'((R\,s')^T\right] \varpi \\
&= \varpi \cdot R \left[\parallel s' \parallel^2 1_{\mathbb{R}^3} - s' s'^T\right] R^T \varpi \ .
\end{aligned}$$

Introducing it into the last term of 5.39 and owing to 5.31 and 5.32 achieves the proof. ■

5.3 Generalized equations of motion

5.3.1 Resultant torsor of the other forces

As in Subsection 3.3.2, we suppose given some linear map:

$$dX \mapsto d\tilde{P} = \tilde{\Gamma}(dX) \ .$$

By analogy with that was done in Chapter 4 with the torsor of internal efforts, we introduce the **covariant differential** of the dynamical torsor defined as:

$$\mathbf{d}\tilde{\mu} = d(\tilde{P}\,\tilde{\mu}'\tilde{P}^T)|_{X'=X} = (\tilde{P}\,d\tilde{\mu}'\tilde{P}^T + d\tilde{P}\,\tilde{\mu}'\tilde{P}^T + \tilde{P}\,\tilde{\mu}'d\tilde{P}^T)\,|_{X'=X} \ .$$

$$\mathbf{d}\tilde{\mu} = (\tilde{P}\,d\tilde{\mu}'\tilde{P}^T + \tilde{\Gamma}(dX)\,\tilde{\mu}'\tilde{P}^T + \tilde{P}\,\tilde{\mu}'(\tilde{\Gamma}(dX))^T)\,|_{X'=X} \ .$$

When X' approaches X, $\tilde{\mu}'$ approaches $\tilde{\mu}$ and $\tilde{P}$ approaches the identity:

$$\mathbf{d}\tilde{\mu} = d\tilde{\mu} + \tilde{\Gamma}(dX)\,\tilde{\mu} + \tilde{\mu}\,(\tilde{\Gamma}(dX))^T \ . \tag{5.40}$$

Taking into account the structure of the dynamical torsor 3.1 and:

$$d\tilde{P} = d\begin{pmatrix} 1 & 0 \\ C & P \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \Gamma_A(dX) & \Gamma(dX) \end{pmatrix} \ , \tag{5.41}$$

where Γ is a Galilean gravitation and Γ_A is a new object that will be further studied, the covariant differential of the dynamical torsor reads:

$$\mathbf{d}\tilde{\mu} = \begin{pmatrix} 0 & \mathbf{d}T^T \\ -\mathbf{d}T & \mathbf{d}J \end{pmatrix} \ ,$$

with:

$$\mathbf{d}T = dT + \Gamma(dX)\,T, \qquad \mathbf{d}J = dJ + \Gamma(dX)\,J + J\,(\Gamma(dX))^T + \Gamma_A(dX)\,T^T - T\,(\Gamma_A(dX))^T \ , \tag{5.42}$$

where the first relation is nothing else 3.37. Dividing by dt and using the same notation as in 3.43, the covariant derivative of the dynamical torsor reads:

$$\mathring{\tilde{\mu}} = \begin{pmatrix} 0 & \mathring{T}^T \\ -\mathring{T} & \mathring{J} \end{pmatrix} , \tag{5.43}$$

with:

$$\mathring{T} = \dot{T} + \Gamma(U)\, T, \qquad \mathring{J} = \dot{J} + \Gamma(U)\, J + J\, (\Gamma(U))^T + \Gamma_A(U)\, T^T - T\, (\Gamma_A(U))^T \tag{5.44}$$

On the other hand, let us introduce the following definition:

Definition 5.10 *The* ***resultant torsor of the other forces*** *(i.e. different from the gravitation) is represented by:*

$$\tilde{\mu}^\star = \begin{pmatrix} 0 & H^T \\ -H & G \end{pmatrix} , \tag{5.45}$$

where the component $H \in \mathbb{R}^4$ represents the resultant of the other forces and is given by 3.75 (or 3.83 for the special case of thrust). $G \in \mathbb{M}_{44}^{skew}$ represent the resultant moment of the other forces.

Thus we can generalized the law 3.11 to the rigid bodies:

Law 5.11 *For any rigid body subjected to a Galilean gravitation and other forces, the motion is governed by the equation:*

$$\mathring{\tilde{\mu}} = \tilde{\mu}^\star .$$

5.3.2 Transformation laws

Once again, guided by Galileo's principle of relativity 1.13, we claim this law is the same in all the Galilean coordinate systems. Thus in another Galilean coordinate system X', we must have:

$$\mathbf{d}\tilde{\mu}' = d\tilde{\mu}' + \tilde{\Gamma}'(dX')\, \tilde{\mu}' + \tilde{\mu}'\, (\tilde{\Gamma}'(dX'))^T .$$

Introducing 3.30 into 5.40, differentiating the products and taking into account 3.28 gives:

$$\mathbf{d}\tilde{\mu} = \tilde{P} \mathbf{d}\tilde{\mu}' \tilde{P}^T , \tag{5.46}$$

provided:

$$\tilde{\Gamma}'(dX') = \tilde{P}^{-1}(\tilde{\Gamma}(\tilde{P}\, dX')\, \tilde{P} + d\tilde{P}) . \tag{5.47}$$

Dividing both members of 5.46 by $dt = dt'$ leads to:

$$\mathring{\tilde{\mu}} = \tilde{P}\, \mathring{\tilde{\mu}}'\, \tilde{P}^T . \tag{5.48}$$

To be consistent with Galileo's principle of relativity 1.13, accounting for 5.48, $\tilde{\mu}^\star$ must be transformed under a change of Galilean coordinate systems $X' \mapsto X$ according to the transformation law 3.2 of a torsor, that justifies the definition 5.10. Using the equivalent formulae 3.30 and the decomposition 5.45, we obtain:

$$H = P\,H', \qquad G = P\,G' P^T + C(P\,H')^T - (P\,H')\,C^T \ , \tag{5.49}$$

where the first relation is nothing else 3.76. For H of the form 3.75 and G of the form:

$$G = \begin{pmatrix} 0 & 0 \\ 0 & -j(M) \end{pmatrix} , \tag{5.50}$$

applying 5.49 with a translation k and a rotation R leaves the null components of H and G while we recover the transformation law 2.10 of the force torsor. In fact, the resultant torsor $\tilde{\mu}^\star$ of the other forces is nothing else the expansion of their resultant static torsor $\breve{\mu}$ when recovering the extra dimension of time. The reader can also verify the relevancy of definition 5.10 to model rocket thrust. With respect to a Galilean coordinate system X' in which the rocket of mass m is at rest, H is given by 3.83 and G is null. In a coordinate system X obtained from X' by a boost v and a translation r, the reader can verify that the dynamical torsor of the thrust is given by 3.84 and:

$$G' = \begin{pmatrix} 0 & -\dot{m} r^T \\ \dot{m} r & -j(r \times (\dot{m}(v+w))) \end{pmatrix} , \tag{5.51}$$

where w is the velocity of the exhaust gases with respect to a Galilean coordinate system X'.

Let us examine now in more detail the transformation law 5.47 of $\tilde{\Gamma}$. Taking into account the decomposition 1.16 of $\tilde{P}$ and 5.41 of $\tilde{\Gamma}$ allows recovering the transformation law 3.49 of the gravitation and revealing the one of the new object Γ_A:

$$\Gamma'_A(dX') = P^{-1}(\Gamma_A(P\,dX') + dC + \Gamma(P\,dX')C) \ . \tag{5.52}$$

Unlike the gravitation Γ, the new object Γ_A has no deep physical meaning but we can link it to physical features of the motion according to the following reasoning. As a rigid body can be considered as a particle of mass $m_\mathcal{B}$ and spin $l_{0\mathcal{B}}$, let us consider two proper coordinate systems X' and $\bar{X}'$ of this particle (defined in Subsection 3.1.2). As the change of coordinate systems $X' \mapsto \bar{X}' = P\,X'$ is linear, the translation C vanishes and:

$$\Gamma'_A(dX') = P^{-1}\,\bar{\Gamma}'_A(P\,dX') \ . \tag{5.53}$$

As the map Γ'_A is linear, the most simple choice for the proper coordinate systems is the identity:

$$\Gamma'_A(dX') = dX' \ ,$$

for which 5.53 is the transformation law for the components of the vector $\overrightarrow{\mathbf{dX'}}$. Next we can deduce the expression of Γ_A in any other coordinate system $X = P\,X' + C$

thanks to its transformation law 5.52 which reads by inversion:

$$\Gamma_A(dX) = P\,\Gamma'_A(dX') - (dC + \Gamma(dX)\,C) = P\,dX' - (dC + \Gamma(dX)\,C)\ ,$$

or in short:

$$\boxed{\Gamma_A(dX) = dX - \mathbf{d}C\ .} \tag{5.54}$$

5.3.3 Equations of motion of a rigid body

Let us consider a proper coordinate system X' of the rigid body considered as a particle. In another coordinate system $X = PX' + C$ obtained by applying a Galilean boost $\dot{r}_\mathcal{B}$ and a translation of the origin $r = 0$ at $k = r$ (hence $\tau = 0$ and $R = 1_{\mathbb{R}^3}$), the dynamical torsor is given by König's first theorem 5.6. In the coordinate system:

$$X = \begin{pmatrix} t \\ r_\mathcal{B} \end{pmatrix}$$

Owing to expression 3.38 of the Galilean gravitation, Γ_A is calculated by the transformation law 5.54:

$$\Gamma_A = d\begin{pmatrix} t \\ r_\mathcal{B} \end{pmatrix} - d\begin{pmatrix} 0 \\ r_\mathcal{B} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ j(\Omega)\,dr_\mathcal{B} - g\,dt & j(\Omega)\,dt \end{pmatrix}\begin{pmatrix} 0 \\ r_\mathcal{B} \end{pmatrix} = \begin{pmatrix} dt \\ -\Omega \times r_\mathcal{B}\,dt \end{pmatrix}\ .$$

Taking into account 5.43 and 5.45, the law 5.11 of the rigid body motion reads:

$$\mathring{T} = H, \qquad \mathring{J} = G\ . \tag{5.55}$$

The first equation is nothing else the law 3.11 of the linear momentum that has been studied in Section 3.5.1. With the notations concerning the rigid body, the equation of motion 3.77 reads:

$$\dot{m}_\mathcal{B} = 0, \qquad \dot{p}_\mathcal{B} = m_\mathcal{B}\,(g - 2\,\Omega \times \dot{r}_\mathcal{B}) + F\ . \tag{5.56}$$

Let us detail now the second one. With some abusive notations again, we write:

$$\mathring{J} = \begin{pmatrix} 0 & -\mathring{q}_\mathcal{B}^T \\ \mathring{q}_\mathcal{B} & -j(\mathring{l}_\mathcal{B}) \end{pmatrix}\ .$$

Also notice that:

$$U = \begin{pmatrix} 1 \\ \dot{r}_\mathcal{B} \end{pmatrix}, \qquad \Gamma_A(U) = \begin{pmatrix} 1 \\ -\Omega \times r_\mathcal{B} \end{pmatrix}\ .$$

Owing to 3.44, the second relation of 5.44 leads to:

$$\mathring{q}_\mathcal{B} = \dot{q}_\mathcal{B} + \Omega \times (q_\mathcal{B} - m_\mathcal{B}\,r_\mathcal{B}) - p_\mathcal{B}\ ,$$

$$\mathring{l}_{\mathcal{B}} = \dot{l}_{\mathcal{B}} + \Omega \times l_{\mathcal{B}} + q_{\mathcal{B}} \times (\Omega \times \dot{r}_{\mathcal{B}} - g) - (\Omega \times r_{\mathcal{B}}) \times p_{\mathcal{B}} \ .$$

Applying Jacobi's identity 12.15 to the last term of the expression of $\mathring{l}_{\mathcal{B}}$, owing to the definition 3.13 of the spin and the expression of the passage $q_{\mathcal{B}} = m_{\mathcal{B}} r_{\mathcal{B}}$ stemming from König's first theorem 5.6, we obtain:

$$\mathring{q}_{\mathcal{B}} = \dot{q}_{\mathcal{B}} - p_{\mathcal{B}}, \qquad \mathring{l}_{\mathcal{B}} = \dot{l}_{\mathcal{B}} + \Omega \times l_{0\mathcal{B}} - q_{\mathcal{B}} \times (g - 2\,\Omega \times \dot{r}_{\mathcal{B}}) \ . \tag{5.57}$$

Once again using the expression of the passage, the second equation in 5.55 reveals –in addition to 5.56– new motion equations:

$$\boxed{\dot{q}_{\mathcal{B}} = p_{\mathcal{B}} \ ,} \tag{5.58}$$

$$\boxed{\dot{l}_{\mathcal{B}} + \Omega \times l_{0\mathcal{B}} = r_{\mathcal{B}} \times m_{\mathcal{B}}(g - 2\,\Omega \times \dot{r}_{\mathcal{B}}) + M \ .} \tag{5.59}$$

The first relation is obvious taking into account the expressions of the passage and the linear momentum given by König's first theorem 5.6, and the time independance of the mass 5.56. The second one is relevant for the study of the motion of the body around it. For the particular case of no spinning ($\Omega = 0$), we deduce the **theorem of the angular momentum**:

Theorem 5.12 *For a rigid body subjected to gravity g without spinning and other forces of resultant moment M, the time derivative of the angular momentum is equal to the resultant moment of the gravity and the other forces:*

$$\dot{l}_{\mathcal{B}} = r_{\mathcal{B}} \times m_{\mathcal{B}} g + M \ .$$

5.4 Motion of a free rigid body around it

Let us consider a rigid body free of gravitation and other forces. A typical application is a satellite at so large distance of the Earth that the gravitation effects are negligeable in a suitable space-time window as discussed in Subsection 3.2.2. Within this window, the mass-centre is in uniform straight motion. For convenience, we use an Eulerian coordinate system in which it is at rest:

$$\dot{r}_{\mathcal{B}} = 0 \ , \tag{5.60}$$

hence the angular momentum given by König's first theorem 5.6 is reduced to the spin:

$$l_{\mathcal{B}} = l_{0\mathcal{B}} \ .$$

We hope to study the motion of the body around it, starting from the new equation of motion 5.59. Because $g = \Omega = M = 0$ in the Eulerian representation, it reads:

$$\dot{l}_{\mathcal{B}} = \dot{l}_{0\mathcal{B}} = 0 \ ,$$

Owing to 5.30 and 5.31, we obtain three integral of the motion:

$$R(t)\mathcal{J}'_{\mathcal{B}}(R(t))^T \varpi(t) = l_{0\mathcal{B}} = C^{te} \ .$$

It is worth pull-backing the spin onto the body at rest by working in the Lagrangian representation but we must take care that the transformation law 3.51 leads to a non vanishing spinning:

$$\Omega' = R^T \varpi = \varpi' \ .$$

where we use 5.6. Thus, in the Lagrangian representation, the equation of motion 5.59 leads to **Euler's equation of motion** of a rigid body:

$$\dot{l}'_{0\mathcal{B}} + \varpi' \times l'_{0\mathcal{B}} = 0 \ . \tag{5.61}$$

Because of the transformation law 3.14 of the spin, 5.30, 5.31 and 5.6:

$$l'_{0\mathcal{B}} = R^T l_{0\mathcal{B}} = R^T \mathcal{J}_{\mathcal{B}} \varpi = R^T R \mathcal{J}'_{\mathcal{B}} R^T \varpi = \mathcal{J}'_{\mathcal{B}} \varpi'$$

Because the moment of inertia matrix $\mathcal{J}'_{\mathcal{B}}$ is time independent, the equation of motion becomes:

$$\mathcal{J}'_{\mathcal{B}} \dot{\varpi}' + \varpi' \times (\mathcal{J}'_{\mathcal{B}} \varpi') = 0 \ .$$

Performing the dot product by ϖ' gives:

$$\varpi' \cdot (\mathcal{J}'_{\mathcal{B}} \dot{\varpi}') = 0 \ , \tag{5.62}$$

and we obtain a new integral of the motion:

$$\varpi' \cdot (\mathcal{J}'_{\mathcal{B}} \varpi') = C^{te} = 2\, e_{\mathcal{B}} \ , \tag{5.63}$$

where, according to König's second theorem 5.9 and 5.60, $e_{\mathcal{B}}$ is the kinetic energy of the body. The geometrical interpretation of this relation –due to Poinsot– is that Poisson's vector ϖ' lies on an ellipsoid of equation 5.63 (figure 5.1). Because the moment of inertia matrix is symmetric, 5.62 reads:

$$\dot{\varpi}' \cdot (\mathcal{J}'_{\mathcal{B}} \varpi') = 0 \ ,$$

thus the tangent plane at ϖ' to **Poinsot's ellipsoid** is perpendicular to the spin angular momentum $l'_{0\mathcal{B}}$. Its distance to the origin:

$$\varpi' \cdot \frac{l'_{0\mathcal{B}}}{\| \, l'_{0\mathcal{B}} \, \|} = \frac{2\, e_{\mathcal{B}}}{\| \, l'_{0\mathcal{B}} \, \|} \ ,$$

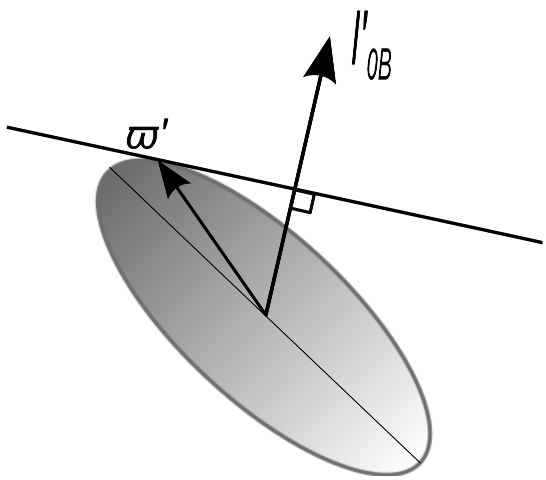

Figure 5.1: Poinsot's construction

is time independent. Pulling back this construction into the Eulerian coordinates, the motion is described by saying that the ellipsoid rolls on the invariable plane drawn perpendicular to the time independent vector $l_{0\mathcal{B}}$ at the invariable distance $2\,e_{\mathcal{B}}/\parallel l'_{0\mathcal{B}}\parallel$ of the origin. The vector drawn from the origin to the contact point is Poisson's vector ϖ'. The curve traced by this point of contact on the ellipsoid and the plane are respectively called the polhode and the herpolhode.

Let us examine now the particular case of a body with rotational symmetry around an axis. In the principal axis of inertia with the third one being the rotational symmetry axis, $A = B$ and the spin angular momentum reads:

$$l'_{0\mathcal{B}} = \mathcal{J}'_{\mathcal{B}}\varpi' = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} \varpi'_1 \\ \varpi'_2 \\ \varpi'_3 \end{pmatrix} = \begin{pmatrix} A\varpi'_1 \\ A\varpi'_2 \\ C\varpi'_3 \end{pmatrix} .$$

Hence Euler's equation of motion 5.61 reads:

$$A\dot{\varpi}'_1 + (C - A)\,\varpi'_2\varpi'_3 = 0 \;,$$

$$A\dot{\varpi}'_2 + (C - A)\,\varpi'_3\varpi'_1 = 0 \;,$$

$$C\dot{\varpi}'_3 = 0 \;.$$

where A and C are time independent. From the last equation, we find a first integral of the motion, the **spin** ϖ'_3 of the body around its rotational symmetry axis. Next, adding the first equation multiplied by ϖ'_1 and the second one multiplied by ϖ'_2 gives the second integral of the motion:

$$\varpi'^2_1 + \varpi'^2_2 = C^{te} \;,$$

and, consequently:

$$\parallel \varpi' \parallel^2 = \varpi'^2_1 + \varpi'^2_2 + \varpi'^2_3 = C^{te} \;.$$

The herpolhode is a circle of time independent radius:

$$\varrho = \sqrt{\| \varpi' \|^2 - \left(\frac{2\, e_{\mathcal{B}}}{\| l'_{0\mathcal{B}} \|} \right)^2} \ .$$

5.5 Motion of a rigid body with a contact point (Lagrange's top)

The toy spinning top is a solid of revolution subjected to the gravity and placed in contact with a horizontal plane (figure 5.2). In absence of spinning, the top naturally tumbles because of the gravity but if it is set spinning about its axis of revolution at a sufficient rotation velocity, it stands upright. In this Section, we would like to explain why the spinning motion prevents the top taking a tumble. For this aim, the top is modelized as a rigid body subjected in a suitable Eulerian coordinate system r to a uniform gravity g without spinning ($\Omega = 0$), in punctual contact with a rough surface at a point $\boldsymbol{O}$ then no sliding is allowed. For convenience, let us pick z's axis directed vertically upward, as determined by a plumb line, and the origine at $\boldsymbol{O}$. Drawing the free body diagram of the top, we remove the support at $\boldsymbol{O}$ and we draw a reaction force F (figure 5.2) acting at $\boldsymbol{O}$. As the gravity is uniform, we can modelize its action upon the top by its resultant $m_{\mathcal{B}} g$ acting at the mass-centre $\boldsymbol{G}$, and obtained thanks to the equation of motion 5.56 in absence of spinning:

$$F = \dot{p}_{\mathcal{B}} - m_{\mathcal{B}}\, g \ ,$$

after determining the trajectory. For this aim, we use the equation of the motion 5.59 of the body around it. As the contact point $\boldsymbol{O}$ is at rest in the considered Eulerian coordinate system, taking into account 5.4, the no sliding condition reads:

$$\dot{r}_{\mathcal{B}} = \varpi \times r_{\mathcal{B}} \ .$$

Hence, starting from the expression of the angular momentum in König's first theorem 5.6 and owing to 12.10, it holds:

$$\begin{aligned} l_{\mathcal{B}} &= m_{\mathcal{B}}\, r_{\mathcal{B}} \times (\varpi \times r_{\mathcal{B}}) + \mathcal{J}_{\mathcal{B}} \varpi = (m_{\mathcal{B}}\, j(r_{\mathcal{B}}) j(r_{\mathcal{B}}) + \mathcal{J}_{\mathcal{B}}) \varpi \\ &= (m_{\mathcal{B}}(\| r_{\mathcal{B}} \|^2\, 1_{\mathbb{R}^3} - r_{\mathcal{B}} r_{\mathcal{B}}^T) + \mathcal{J}_{\mathcal{B}}) \varpi \ . \end{aligned} \tag{5.64}$$

By Huygens' theorem 5.8, the angular momentum of the top reads:

$$l_{\mathcal{B}} = \mathcal{J}_{\mathcal{B}O} \varpi \ . \tag{5.65}$$

In a similar way, starting from the expression of the kinetic energy in König's second theorem 5.9 and owing to 12.16, it holds:

$$e_{\mathcal{B}} = \frac{1}{2} m_{\mathcal{B}} (\| r_{\mathcal{B}} \|^2 \| \varpi \|^2 - (r_{\mathcal{B}} \cdot \varpi)^2) + \frac{1}{2} \varpi \cdot (\mathcal{J}_{\mathcal{B}}\, \varpi) \ .$$

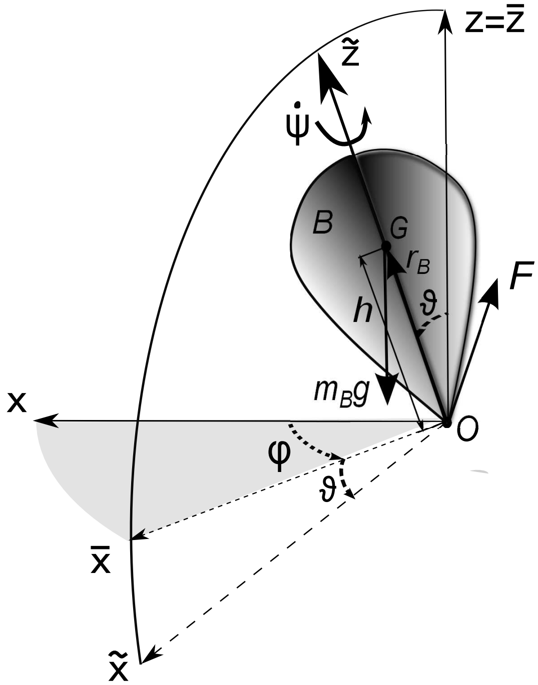

Figure 5.2: Lagrange's top

Because of Huygens' theorem 5.8, the kinetic energy of the top reads:

$$e_{\mathcal{B}} = \frac{1}{2}\,\varpi \cdot (\mathcal{J}_{\mathcal{B}O}\varpi)\ . \tag{5.66}$$

On this ground, we consider a Lagrangian coordinate system r' with origin at the contact point $\boldsymbol{O}$, the rotational symmetry axis being the one of z', and we modelize the body motion by the rotation matrix R of the map from the reference Eulerian coordinate system r onto $r' = R^T r$. To express ϖ in terms of Euler's angles, we differentiate 3.20 with respect to the time:

$$\dot{R} = \dot{R}_\varphi R_\vartheta R_\psi + R_\varphi \dot{R}_\vartheta R_\psi + R_\varphi R_\vartheta \dot{R}_\psi\ .$$

Hence, we have:

$$\dot{R}R^T = \dot{R}_\varphi R_\varphi^T + R_\varphi(\dot{R}_\vartheta R_\vartheta^T)R_\varphi^T + (R_\varphi R_\vartheta)(\dot{R}_\psi R_\psi^T)(R_\varphi R_\vartheta)^T\ .$$

ϖ_φ, ϖ_ϑ and ϖ_ψ being respectively Poisson's vectors of R_φ, R_ϑ and R_ψ, it holds, owing to 12.20:

$$j(\varpi) = j(\varpi_\varphi) + j(R_\varphi \varpi_\vartheta) + j(R_\varphi R_\vartheta \varpi_\psi)\ .$$

Because the map j is linear and regular, we obtain:

$$\varpi = \varpi_\varphi + R_\varphi \varpi_\vartheta + R_\varphi R_\vartheta \varpi_\psi\ ,$$

Introducing $R_{\varphi\vartheta} = R_\varphi R_\vartheta$, its pull back onto the coordinate system $\tilde{r} = R^T_{\varphi\vartheta} r$ is:

$$\tilde{\varpi} = R^T_\vartheta R^T_\varphi \varpi = R^T_{\varphi\vartheta}\varpi_\varphi + R^T_\vartheta \varpi_\vartheta + \varpi_\psi \ . \tag{5.67}$$

Taking into account 3.21, it holds:

$$\tilde{\varpi} = \begin{pmatrix} \cos\varphi\cos\vartheta & \sin\varphi\cos\vartheta & -\sin\vartheta \\ -\sin\varphi & \cos\varphi & 0 \\ \cos\varphi\sin\vartheta & \sin\varphi\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} + \begin{pmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\vartheta} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} ,$$

and we obtain:

$$\tilde{\varpi} = \begin{pmatrix} -\dot{\varphi}\sin\vartheta \\ \dot{\vartheta} \\ \dot{\psi} + \dot{\varphi}\cos\vartheta \end{pmatrix} . \tag{5.68}$$

According to 5.37, the moment of inertia matrix reads in the Lagrangian coordinate system $\tilde{r}$:

$$\tilde{\mathcal{J}}_{\mathcal{B}O} = R_\psi \mathcal{J}'_{\mathcal{B}O} R^T_\psi \ ,$$

or, owing to 3.21:

$$\tilde{\mathcal{J}}_{\mathcal{B}O} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

that leads to:

$$\tilde{\mathcal{J}}_{\mathcal{B}O} = \mathcal{J}'_{\mathcal{B}O} \tag{5.69}$$

which is nothing else an expression of the rotational symmetry of the body around the $\tilde{z}$'s axis. By putting $\varpi_\psi = 0$ in 5.67, Poisson's vector of the rotation matrix $R_{\varphi\vartheta}$ is found to be in the Eulerian coordinate system $\tilde{r}$:

$$\tilde{\varpi}_{\varphi\vartheta} = R^T_{\varphi\vartheta}\varpi_\varphi + R^T_\vartheta \varpi_\vartheta = \begin{pmatrix} -\dot{\varphi}\sin\vartheta \\ \dot{\vartheta} \\ \dot{\varphi}\cos\vartheta \end{pmatrix} .$$

Now, we are able to find three integral of the motion:

- Let us calculate the time derivative of the total kinetic energy 5.38 and take into account the equation of motion of the elementary mass $dm(s')$ in the reference Eulerian coordinate system r were the spinning Ω vanishes and the gravity g is uniform:

  $$\dot{e}_{\mathcal{B}} = \iiint_{\mathcal{B}} v \cdot \dot{v}\, dm(s') = \iiint_{\mathcal{B}} v \cdot g\, dm(s') = \left(\iiint_{\mathcal{B}} v\, dm(s') \right) \cdot g \ .$$

  Because of definition 5.5 and König's first theorem 5.6, one has:

  $$\dot{e}_{\mathcal{B}} = p_{\mathcal{B}} \cdot g = m_{\mathcal{B}}\, g \cdot \dot{r}_{\mathcal{B}} \ .$$

Introducing the gravitational potential:

$$\phi = -g \cdot r_{\mathcal{B}} \ ,$$

we obtain a first integral of the motion, the total energy:

$$e_T = e_{\mathcal{B}} + m_{\mathcal{B}} \phi \ .$$

The total kinetic energy 5.66 becomes:

$$e_{\mathcal{B}} = \frac{1}{2} \left(R_{\varphi\vartheta} \tilde{\varpi}\right)^T \cdot \left(\mathcal{J}_{\mathcal{B}O}(R_{\varphi\vartheta} \tilde{\varpi})\right) = \frac{1}{2} \tilde{\varpi} \cdot \left(\tilde{\mathcal{J}}_{\mathcal{B}O} \tilde{\varpi}\right) \ ,$$

where $\tilde{\mathcal{J}}_{\mathcal{B}O} = R^T_{\varphi\vartheta} \mathcal{J}_{\mathcal{B}O} R_{\varphi\vartheta}$, according to the transformation law 5.37. Owing to 5.68 and 5.69, one has:

$$e_{\mathcal{B}} = \frac{1}{2} \left(A \, (\dot{\vartheta}^2 + \dot{\varphi}^2 \sin^2 \vartheta) + C \, (\dot{\psi} + \dot{\varphi} \cos \vartheta)^2 \right) \ .$$

Besides, h being the distance between the mass-centre $\boldsymbol{G}$ and the contact point $\boldsymbol{O}$, one has:

$$r_{\mathcal{B}} = R_{\varphi\vartheta} \, \tilde{r}_{\mathcal{B}} = \begin{pmatrix} \cos\varphi \cos\vartheta & -\sin\varphi & \cos\varphi \sin\vartheta \\ \sin\varphi \cos\vartheta & \cos\varphi & \sin\varphi \sin\vartheta \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} = h \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix} ,$$

and the gravitational potential reads:

$$\phi = - \begin{pmatrix} 0 & 0 & - \parallel g \parallel \end{pmatrix} h \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix} = \parallel g \parallel \ h \cos\vartheta \ .$$

The first integral of the motion, the kinetic energy reads:

$$e_T = \frac{1}{2} \left(A \, (\dot{\vartheta}^2 + \dot{\varphi}^2 \sin^2 \vartheta) + C \, (\dot{\psi} + \dot{\varphi} \cos \vartheta)^2 \right) + m_{\mathcal{B}} \ \parallel g \parallel \ h \cos\vartheta = C^{te} \ . \tag{5.70}$$

- The second integral of the motion is obtained considering the equation 5.59 of motion in the Eulerian coordinate system $\tilde{r}$:

$$\dot{\tilde{l}}_{\mathcal{B}} + \tilde{\Omega} \times \tilde{l}_{0\mathcal{B}} = \tilde{r}_{\mathcal{B}} \times m_{\mathcal{B}} (\tilde{g} - 2 \, \tilde{\Omega} \times \dot{\tilde{r}}_{\mathcal{B}}) \ ,$$

where the moment of the reaction force acting at the origin $\boldsymbol{O}$ vanishes. Because the position vector $\tilde{r}_{\mathcal{B}}$ of the mass-centre is collinear to the basis vector $e_{\tilde{z}}$, we obtain by projection:

$$e_{\tilde{z}} \cdot \left(\dot{\tilde{l}}_{\mathcal{B}} + \tilde{\Omega} \times \tilde{l}_{0\mathcal{B}} \right) = 0 \ . \tag{5.71}$$

Owing to 5.65, 5.68 and 5.69, the angular momentum reads:

$$\tilde{l}_{\mathcal{B}} = \tilde{\mathcal{J}}_{\mathcal{B}O}\tilde{\varpi} = \begin{pmatrix} -A\dot{\varphi}\sin\vartheta \\ A\dot{\vartheta} \\ C\,(\dot{\psi} + \dot{\varphi}\cos\vartheta) \end{pmatrix} . \tag{5.72}$$

On the other hand, the spin angular momentum reads:

$$\tilde{l}_{0\mathcal{B}} = \tilde{\mathcal{J}}_{\mathcal{B}}\tilde{\varpi}$$

Owing to Huygens' theorem 5.8, namely 5.35, one has:

$$\tilde{\mathcal{J}}_{\mathcal{B}} = \tilde{\mathcal{J}}_{\mathcal{B}O} - m_{\mathcal{B}}(\|\ \tilde{r}_{\mathcal{B}}\ \|^2\ 1_{\mathbb{R}^3} - \tilde{r}_{\mathcal{B}}\tilde{r}_{\mathcal{B}}^T) = \begin{pmatrix} A^* & 0 & 0 \\ 0 & A^* & 0 \\ 0 & 0 & C^* \end{pmatrix} ,$$

where $A^* = A - m_{\mathcal{B}}h^2$, $C^* = C$ and, taking into account 5.68:

$$\tilde{l}_{0\mathcal{B}} = \begin{pmatrix} -A^*\dot{\varphi}\sin\vartheta \\ A^*\dot{\vartheta} \\ C^*\,(\dot{\psi} + \dot{\varphi}\cos\vartheta) \end{pmatrix} .$$

Moreover, using the transformation law 3.51 of the spinning and because $\Omega = 0$ in the reference Eulerian coordinate system r, one has:

$$\tilde{\Omega} = \tilde{\varpi}_{\varphi\vartheta} \ .$$

After calculation, it holds:

$$\dot{\tilde{l}}_{\mathcal{B}} + \Omega \times \tilde{l}_{0\mathcal{B}} = \begin{pmatrix} -\frac{d}{dt}(A\dot{\varphi}\sin\vartheta) + (C^* - A^*)\dot{\vartheta}\dot{\varphi}\cos\vartheta + C^*\dot{\vartheta}\dot{\psi} \\ \frac{d}{dt}(A\dot{\vartheta}) + (C^* - A^*)\dot{\varphi}^2\sin\vartheta\cos\vartheta + C^*\dot{\varphi}\dot{\psi}\sin\vartheta \\ \frac{d}{dt}(C\,(\dot{\psi} + \dot{\varphi}\cos\vartheta)) \end{pmatrix} .$$

Because C is time independent, relation 5.71 leads to a second integral of the motion, the projection of the angular momentum onto the top symmetry axis:

$$\dot{\psi} + \dot{\varphi}\cos\vartheta = C^{te} = n \ , \tag{5.73}$$

often called **spin** and denoted by n. It is worth to remark that, as for the motion of a free rigid body around it, the spin integral of motion results from the rotational symmetry of the body.

- The third integral of motion is obtained considering once again the equation 5.59 of motion but in the reference Eulerian coordinate system r. Hence, the assumptions of the theorem 5.12 of the angular momentum are fulfilled and the equation of motion is reduced to:

$$\dot{l}_{\mathcal{B}} = r_{\mathcal{B}} \times m_{\mathcal{B}}g \ . \tag{5.74}$$

The gravity g being collinear to the time-independent basis vector e_z along the vertical z's axis, it holds:

$$\dot{l}_{\mathcal{B}} \cdot e_z = \frac{d}{dt}(l_{\mathcal{B}} \cdot e_z) = \dot{l}_z = 0 \ ,$$

and the z-component of the angular momentum is an integral of the motion:

$$l_z = C^{te} \ . \tag{5.75}$$

For the change of coordinate system $r = R_{\varphi\vartheta}\tilde{r}$, the velocity of transport 3.29 vanishes at $r = 0$, then the transformation law 3.8 of the angular momentum gives :

$$l_{\mathcal{B}} = R_{\varphi\vartheta}\tilde{l}_{\mathcal{B}} \ .$$

Introducing:

$$\tilde{e}_z = R^T_{\varphi\vartheta} e_z = \begin{pmatrix} \cos\varphi\cos\vartheta & \sin\varphi\cos\vartheta & -\sin\vartheta \\ -\sin\varphi & \cos\varphi & 0 \\ \cos\varphi\sin\vartheta & \sin\varphi\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\vartheta \\ 0 \\ \cos\vartheta \end{pmatrix} \ ,$$

and taking into account 5.72 and 5.73, the last integral of motion reads:

$$l_z = e_z^T l_{\mathcal{B}} = \tilde{e}_z^T \tilde{l}_{\mathcal{B}} = A\dot{\varphi}\sin^2\vartheta + C\,n\cos\vartheta \ . \tag{5.76}$$

Setting aside definitely the events $C = 0$ and $n = 0$, we define the adimensional constants:

$$\alpha = \frac{2e_T - C\,n^2}{2\,m_{\mathcal{B}}\ \|\ g\ \|\ h}, \qquad \beta = \frac{l_z}{C\,n}, \qquad b = \frac{C}{A}, \qquad n_* = n\sqrt{\frac{A}{2\,m_{\mathcal{B}}\ \|\ g\ \|\ h}} \ ,$$

and the adimensional variables:

$$x = \cos\vartheta, \qquad \varphi, \qquad \psi, \qquad \tau = \sqrt{\frac{2\,m_{\mathcal{B}}\ \|\ g\ \|\ h}{A}}\, t \ .$$

Denoting $'$ the derivative with respect to τ, the integral of the motion 5.70, 5.73 and 5.76 read:

$$x'^2 + \varphi'^2(1-x^2)^2 = (\alpha - x)\,(1-x^2) \ ,$$

$$\psi' + \varphi' x = n_* \ ,$$

$$\varphi'(1-x^2) = b\,n_*(\beta - x) \ .$$

Eliminating φ', we get the differential equation:

$$x'^2 = f(x) = (\alpha - x)\,(1-x^2) - b^2 n_*^2(\beta - x) \ .$$

The motion is possible only if $f(x)$ is positive and $\mid x \mid \leq 0$ (since $x = \cos\vartheta$). For large values of $\mid x \mid$, $f(x)$ is dominated by the cubic term in x, then

$$\lim_{x\to\pm\infty} f(x) = \pm\infty \ .$$

As $f(-1) < 0$ and $f(1) < 0$, the cubic polynomial f has three real zeros x_1, x_2, x_3 such that:

$$-1 < x_1 < x_2 < 1 < x_3 \ ,$$

special cases of equality being disregarded here. The variable x oscillates within the interval $[x_1, x_2]$. Then the nutation angle ϑ is varrying between limit values corresponding to x_1 and x_2, that explains why the spinning top does not take a tumble. The azimutal angle φ is given by:

$$\varphi' = \frac{b\, n_*(\beta - x)}{1 - x^2} \ .$$

It is clear that φ' has one sign throughout the motion if and only if β lies outside the interval $[x_1, x_2]$. The motion is most clearly followed by tracing the path of $e_{\tilde{z}}$ on the unit sphere with coordinates ϑ, φ. This path is bounded by the two circles $x = x_1$ (above) and $x = x_2$ (below), and the path crosses itself if and only if φ' changes sign during the motion.

5.6 Comments for experts

[Comment 1]This partial analogy is a cause of misleading and confusion in literature where co-torsors are erroneously identified to torsors.

Chapter 6

Calculus of Variations

6.1 Introduction

Since a long time, it has been remarked that the laws of many natural phenomena can be obtained by realizing an extremum (minimum or maximum) of a certain physical quantity assigned to a function modelizing the considered phenomenon. In other words, we use a real-valued map defined on a set of functions and called a functional (a function of functions). Such a formulation, called a variational principle, is used to obtain the physical laws of the consider phenomenon by means of the calculus of variations. The variational principles have over all a mnemonic value which allows deducing the physical laws in a consistent and systematic way.

In the Dynamics of particles and rigid bodies, the considered functional is called the action and the corresponding variational principle is the principle of least action that allows to deduce the equations of motion in a more abstract way as in Chapters 3 and 5.

The starting point is the **Lagrangian**, *i.e.* a differentiable real function:

$$\mathcal{L} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} : (t, y, z) \longmapsto \lambda = \mathcal{L}(t, y, z) \ .$$

To a differentiable map $t \longmapsto y$ we associate the number:

$$\alpha[y] = \int_{t_0}^{t_1} \mathcal{L}(t, y, \dot{y}) \, dt \ ,$$

called the **action**, that defines the **functional** $y \longmapsto \alpha[y]$. Let us suppose that y depends on a parameter ϵ so that the map:

$$\begin{pmatrix} t \\ \epsilon \end{pmatrix} \longmapsto y \ , \tag{6.1}$$

is twice continuously differentiable and:

$$\forall \epsilon, \qquad y(t_0, \epsilon) = y_0, \qquad y(t_1, \epsilon) = y_1 \ . \tag{6.2}$$

The action depends now on ϵ:

$$\alpha(\epsilon) = \alpha[y].$$

if the action has a minimum for the map $t \longmapsto y(t,0)$, the function $\epsilon \longmapsto \alpha$ has a minimum at $\epsilon = 0$ and thus:

$$\delta\alpha = \alpha'(0) = \int_{t_0}^{t_1} \left.\frac{\partial \mathcal{L}}{\partial \epsilon}\right|_{\epsilon=0} dt = 0 \ . \tag{6.3}$$

The time derivative $\dot{y}$ is now a simplified notation for the partial derivative of y with respect to t and the partial derivative with respect to the parameter ϵ:

$$\delta y = \left.\frac{\partial y}{\partial \epsilon}\right|_{\epsilon=0} ,$$

is called the **variation** of y, hence the name of **calculus of variations**. As the Lagrangian depends on ϵ through y and $\dot{y}$, the chain rule provides:

$$\frac{\partial \mathcal{L}}{\partial \epsilon} = \frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{\partial^2 y}{\partial \epsilon \partial t} = \frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{\partial^2 y}{\partial t \partial \epsilon} ,$$

because y is twice continuously differentiable. With simplified notations, one has:

$$\left.\frac{\partial \mathcal{L}}{\partial \epsilon}\right|_{\epsilon=0} = \frac{\partial \mathcal{L}}{\partial y}\,\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\,\delta\left(\frac{dy}{dt}\right) = \frac{\partial \mathcal{L}}{\partial y}\,\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{d}{dt}(\delta y) \ , \tag{6.4}$$

where the derivative symbols d/dt and δ are permuted. Substituting this expression into the variation 6.3 leads to:

$$\delta\alpha = \int_{t_0}^{t_1} \left[\frac{\partial \mathcal{L}}{\partial y}\,\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{d}{dt}(\delta y)\right] dt = 0 \ .$$

Integrating by parts, it holds:

$$\delta\alpha = \left[\frac{\partial \mathcal{L}}{\partial \dot{y}}\,\delta y\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)\right] \delta y \, dt = 0 \ .$$

Taking into account 6.2, the variation of y vanishes at $t = t_0$ and $t = t_1$ and we have for any map 6.1 hence for any variation δy:

$$\delta\alpha = \int_{t_0}^{t_1} \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)\right] \delta y \, dt = 0 \ . \tag{6.5}$$

We are in situation where f being a given map defined on $[t_0, t_1]$ we have:

$$\int_{t_0}^{t_1} f(t)\,\delta y(t)\, dt = 0 \ ,$$

for every continuous function $t \longmapsto \delta y$, for instance $\delta y = f\, g$ where $g(t) = -(t - t_0)\,(t - t_1)$:

$$\int_{t_0}^{t_1} f(t)\, \delta y(t)\, dt = \int_{t_0}^{t_1} (f(t))^2\, g(t)\, dt = 0 \ .$$

The integrand is non negative so it must be zero. Applying this result to 6.5 leads to:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = 0 \ , \tag{6.6}$$

and by transposition to **Euler-Lagrange equations**:

$$\boxed{\frac{d}{dt}\,(grad_{\dot{y}}\, \mathcal{L}) - grad_y\, \mathcal{L} = 0 \ .} \tag{6.7}$$

The curve of $\mathbb{R}^n$ represented by the map $t \mapsto y$ realizing the minimum of the action is called **natural path**. Taking into account (6.6), we have along the natural path:

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{d\dot{y}}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)\dot{y} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{d\dot{y}}{dt} \ ,$$

that reads:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\,\dot{y} - \mathcal{L}\right) = 0 \ ,$$

hence we obtain a quantity preserved along the natural path, Legendre's transform of the Lagrangian and called **Hamiltonian**:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{y}}\,\dot{y} - \mathcal{L} = C^{te} \tag{6.8}$$

Next, let us examine current situations in which the variable y is subjected to a constraint:

$$\forall t \in [t_0, t_1], \qquad f(t, y(t)) = 0 \ .$$

We would like that:

$$\{\delta f = grad_y\, f \cdot \delta y = 0\} \qquad \Rightarrow \qquad \left\{\delta \mathcal{L} = \left[\frac{d}{dt}\,(grad_{\dot{y}}\, \mathcal{L}) - grad_y\, \mathcal{L}\right] \cdot \delta y = 0\right\} \ ,$$

hence there exists a Lagrange's multiplier $\lambda \in \mathbb{R}$ such that:

$$\boxed{\frac{d}{dt}\,(grad_{\dot{y}}\, \mathcal{L}) - grad_y\, \mathcal{L} = \lambda\, grad_y\, f \ .} \tag{6.9}$$

6.2 Particle subjected to the Galilean gravitation

6.2.1 Guessing the Lagrangian expression

We would like to find the equations of motion of a particle by this method. Although at first glance it may seem attractive, to be honest we have to say that it is not so easy to know the expression of the corresponding Lagrangian. To get it, we use an heuristic way. To lay the ground, we first try to guess it in the simple case of a free particle, hence in uniform straight motion in some Galilean coordinate system X'. The studied phenomenon is the trajectory of a particle of mass m, modelized by the function $t \longmapsto r'$. In absence of gravitation, the equation of motion 3.45 is reduced to:

$$\frac{dp'}{dt} = 0 \ .$$

From the comparison with Euler-Lagrange equations 6.7 (where y is r' and $\dot{r}'$ is v') we gather:

$$grad_{v'} \, \mathcal{L} = p' = m \, v', \qquad grad_{r'} \, \mathcal{L} = 0 \ .$$

Modulo a constant, an obvious solution is:

$$\mathcal{L}(t, r', v') = \frac{1}{2} m \parallel v' \parallel^2 \ ,$$

which is nothing else the kinetic energy 3.73.

Unfortunately, the calculus of variations is littered with traps (but we shall learn to avert some of thems). The problem of this Lagrangian is that it was found in a very peculiar situation where the particle is in uniform straight motion in X'. In fact, the previous expression is not general and our goal now is to find its generic form in any Galilean coordinate system X. For this aim, we use the boost method in the spirit of Section 3.1.2. The coordinate change $X' \longmapsto X$ being characterized by a boost u and a rotation R, we use the velocity addition formula 1.13 to express the Lagrangian in terms the velocity in the new Galilean coordinate system X:

$$\mathcal{L} = \frac{1}{2} m \parallel v' \parallel^2 = \frac{1}{2} m \parallel R \, v' \parallel^2 = \frac{1}{2} m \parallel v - u \parallel^2 \ ,$$

thus, expanding:

$$\mathcal{L} = \frac{1}{2} m \parallel v \parallel^2 + \frac{1}{2} m \parallel u \parallel^2 - m \, u \cdot v \ . \tag{6.10}$$

For this new expression of the Lagrangian, Euler-Lagrange equations:

$$\frac{d}{dt} \left(grad_v \, \mathcal{L} \right) - grad_r \, \mathcal{L} = 0 \ , \tag{6.11}$$

give:

$$\frac{d}{dt} \left(m \, (v - u) \right) + m \, (grad_r u) \, (v - u) = 0 \ .$$

Taking into account the expression 3.29 of the velocity of transport, one has:

$$m\,\dot{v} = m\,[\dot{u} + j(\varpi)\,(v-u)]\ ;$$

that, owing to the expression 3.53 of the acceleration of transport, leads to:

$$m\,\dot{v} = m\,a_t\ .$$

In the old coordinate system X', the particle is gravitation free hence $g' = \Omega' = 0$ and, taking into account the transformation law 3.52 of the gravitation, we recover Souriau's equation of motion 3.47:

$$m\ddot{r} = m\,(g - 2\,\Omega \times v)\ ,$$

in any Galilean coordinate system X.

6.2.2 The potentials of the Galilean gravitation

Next, let us consider a more general case where there is not necessary particular Galilean coordinate systems in which the particle is in uniform straight motion. Having a look to the Lagrangian 6.10, we claim that the Lagrangian for a particle subjected to a Galilean gravitation has the following general form:

$$\boxed{\mathcal{L}(t,r,v) = \tfrac{1}{2}\,m \parallel v \parallel^2 + m\,A\cdot v - m\,\phi\ ,} \tag{6.12}$$

where $(r,t) \longmapsto \phi(r,t) \in \mathbb{R}$ and $(r,t) \longmapsto A(r,t) \in \mathbb{R}^3$ are given scalar and vector fields assumed to modelize the gravitation (for instance, the particular Lagrangian 6.10 is obtained with $\phi = - \parallel u \parallel^2 /2$ and $A = -u$). Corresponding Euler-Lagrange equation reads:

$$\frac{d}{dt}\,(grad_v\,\mathcal{L}) - grad_r\,\mathcal{L} = \frac{d}{dt}\,[m\,(v + A)] + m\ [grad\,\phi - (grad\,A)\,v] = 0.$$

As the field A depends on r and t, one has:

$$\dot{p} + m\,\left[\frac{\partial A}{\partial t} + \frac{\partial A}{\partial r}\,v + grad\,\phi - (grad\,A)\,v\right] = 0.$$

$$\dot{p} + m\,\left[grad\,\phi + \frac{\partial A}{\partial t} + j(curl\,A)\,v\right] = 0.$$

We recover the equation of motion (3.45):

$$\dot{p} = m\,(g - 2\,\Omega \times v),$$

as Euler-Lagrange variation equation of the least action principle, provided we put:

$$\boxed{g = -grad\,\phi - \frac{\partial A}{\partial t}, \qquad \Omega = \tfrac{1}{2}\,curl\,A\ .} \tag{6.13}$$

The fields ϕ and A are called the **potentials of the Galilean gravitation**. It is said that the components g, Ω of the Galilean gravitation **admit** or **are generated by the potentials** ϕ, A. In fact, we already know the scalar potential ϕ, we met it in the particular case of the Newtonian gravitation and it was given by 3.72 in Kepler's problem. By a straightforward calculation, the reader can verify that, for a given Galilean gravitation, a necessary condition for the existence of the gravitation potentials is

$$curl\,g + 2\frac{\partial \Omega}{\partial t} = 0, \qquad div\,\Omega = 0\ .$$

It is worth to remark that there is not always a variational formulation because it is conditioned to the satisfaction of these two conditions (this is another trap of the calculus of variation).

Besides, let us remark that for a given gravitation field, the choice of the potentials ϕ and A is not unique. Indeed, let ϕ^* and A^* be potentials allowing to recover the same gravity and spinning fields by 6.13. Hence $\Delta\phi = \phi^* - \phi$ and $\Delta A = A^* - A$ satisfy:

$$grad\,(\Delta\phi) + \frac{\partial}{\partial t}(\Delta A) = 0, \qquad curl\,(\Delta A) = 0\ .$$

Because of the last condition, there exists (at least within a simply connected subdomain of $\mathbb{R}^4$) a scalar field $(r,t) \longmapsto f(r,t)$ such that:

$$\Delta A = grad\,f\ ,$$

and the first condition reads:

$$grad\left(\Delta\phi + \frac{\partial f}{\partial t}\right) = 0\ .$$

We can conclude that:

$$\boxed{\phi^* = \phi - \frac{\partial f}{\partial t}, \qquad A^* = A + grad\,f\ ,} \tag{6.14}$$

leads to the same gravitation field as ϕ and A. The arbitrary field f is called a **gauge function** and the previous condition is the **gauge transformation**.

At this stage, it is worth discussing once again an important integral of the motion already met about Kepler's problem in Section 3.4 but now in a more general context. Inded, along the natural path, *i.e.* the trajectory, the Hamiltonian is constant. Taking into account the expression (6.12) of the Lagrangian, one obtains the integral of the motion:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial v} v - \mathcal{L} = \frac{1}{2} m \parallel v \parallel^2 + m \phi \ , \tag{6.15}$$

which is noting else the total energy (3.74).

Before going further, let us have a look once again to the exemple of subsection 3.3.1. In the Galilean coordinate system X' such that $g' = \Omega' = 0$, the particle is at rest. For the observer rotating at $r_0 = 0$ at constant rotation velocity ϖ and working with the Galilean coordinate system X, the potentials of the gravitation are, under the conditions 3.63, given by:

$$\phi = -\frac{1}{2} \parallel u \parallel^2 = -\frac{1}{2} \parallel \varpi \times r \parallel^2, \qquad A = -u = -\varpi \times r \ .$$

As exercice, the reader can verify that 6.13 allows to recover the expression 3.64 of the spinning and gravity fields.

6.2.3 Transformation law of the potentials of the gravitation

Introducing into the Lagrangian 6.12 the expression of v given by the velocity addition formula 1.13, we obtain after straighforward calculations:

$$\mathcal{L}(t, r', v') = \frac{1}{2} m \parallel v' \parallel^2 - m \phi' + m A' \cdot v' \ ,$$

where:

$$\boxed{\phi' = \phi - A \cdot u - \frac{1}{2} \parallel u \parallel^2, \qquad A' = R^T (A + u) \ .} \tag{6.16}$$

Let us prove it is the **transformation law of the Galilean gravitation potentials** in the following sense:

Theorem 6.1 *If the components g, Ω of the Galilean gravitation in the Galilean coordinate system X are generated by the potentials ϕ, A, according to 6.13, then the corresponding components g', Ω' in another Galilean coordinate system X' admit the potentials ϕ', A' given by 6.16:*

$$g' = -grad_{r'} \phi' - \frac{\partial A'}{\partial t'}, \qquad \Omega' = \frac{1}{2} curl_{r'} A' \ . \tag{6.17}$$

The quantity:

$$I = \phi + \parallel A \parallel^2 / 2 \tag{6.18}$$

is a galilean invariant.

Proof. The calculus is decomposed in four steps.

- **Step 1:** *recasting the transformation law of the gravity.* Let us remark by time differentiating 3.29 at constant r that 3.60 is nothing else:

$$a_t^* = \frac{\partial u}{\partial t} \ .$$

Hence 3.61 reads:

$$\boxed{g = \frac{\partial u}{\partial t} + \varpi \times u + 2\,\Omega \times u + R\,g' \ ,} \tag{6.19}$$

which is a more compact way to write the transformation law of the gravity 3.62.

- **Step 2:** *establishing the transformation law for the derivatives of a column field.* Owing to 3.28, it holds for any column field $X \longmapsto v'(X) \in \mathbb{R}^n$:

$$dv' = \frac{\partial v'}{\partial X}\, dX = \frac{\partial v'}{\partial X}\, P\, dX' \ .$$

On the other hand, v' being seen as a function of X' through the coordinate change $X \longmapsto X'$, one has:

$$dv' = \frac{\partial v'}{\partial X'}\, dX' \ .$$

dX' being arbitrary, we obtain by comparing the previous relations:

$$\frac{\partial v'}{\partial X'} = \frac{\partial v'}{\partial X}\, P \ .$$

Taking into account 1.9, one has:

$$\frac{\partial v'}{\partial t'} = \frac{\partial v'}{\partial t} + \frac{\partial v'}{\partial r}\, u, \qquad \frac{\partial v'}{\partial r'} = \frac{\partial v'}{\partial r}\, R \ . \tag{6.20}$$

- **Step 3:** *demonstrating the second condition 6.17.* Applying this formula to the potential field A and owing to its transformation law 6.16, one has:

$$\frac{\partial A'}{\partial r'} = \frac{\partial}{\partial r}(R^T(A+u))\,R = R^T\,\frac{\partial}{\partial r}(A+u)\,R \ ,$$

because R is independent of r. Owing to the definition 12.40 of *curl* and 12.20, it holds:

$$j(curl_{\,r'}\, A') = \frac{\partial A'}{\partial r'} - \left(\frac{\partial A'}{\partial r'}\right)^T = R^T \left[\frac{\partial}{\partial r}(A+u) - \left(\frac{\partial}{\partial r}(A+u)\right)^T\right] R \ ,$$

$$j(curl_{r'}\,A') = R^T j(curl_r\,(A+u))\,R = j(R^T curl_r\,(A+u))\ .$$

Because the map j is regular, we obtain:

$$curl_{r'}\,A' = R^T(curl_r\,A + curl_r\,u)\ .$$

Differentiating 3.29 with respect to r gives:

$$\frac{\partial u}{\partial r} = j(\varpi)\ , \qquad grad_r\,u = -j(\varpi)\ , \tag{6.21}$$

and using once again 12.40:

$$curl_r\,u = 2\varpi\ , \tag{6.22}$$

that proves, owing to 6.13, the second condition 6.17:

$$\frac{1}{2}\,curl_{r'}\,A' = R^T(\frac{1}{2}\,curl_r\,A + \varpi) = \Omega'\ .$$

- **Step 4:** *demonstrating the first condition 6.17.* Applying the first condition 6.20 to A and taking into account 6.16, one has:

$$\frac{\partial A'}{\partial t'} = \frac{\partial A'}{\partial t} + \frac{\partial A'}{\partial r}\,u = \dot{R}^T(A+u) + R^T\left(\frac{\partial}{\partial t}(A+u) + \frac{\partial}{\partial r}(A+u)\,u\right)\ ,$$

or, taking into account 3.25:

$$\frac{\partial A'}{\partial t'} = R^T\left(\frac{\partial}{\partial t}(A+u) + \frac{\partial}{\partial r}(A+u)\,u - \varpi \times (A+u)\right)\ .$$

On the other hand, applying the second condition 6.20 to ϕ, transposing, owing to 6.16 and 12.36, one has:

$$grad_{r'}\,\phi' = R^T\,(grad_r\,\phi - (grad_r\,(A+u))\,u - (grad_r\,u)\,A)\ .$$

Combining the last two results and owing to 6.13 gives:

$$-grad_{r'}\,\phi' - \frac{\partial A'}{\partial t'} = R^T\left[g - \frac{\partial u}{\partial t} + \left(grad_r\,(A+u) - \frac{\partial}{\partial r}(A+u)\right)\,u + (grad_r\,u)\,A + \varpi \times (A+u)\right]\ .$$

Owing to the definition 12.40 of *curl*, 6.13, 6.22 and 6.21, one has:

$$-grad_{r'}\,\phi' - \frac{\partial A'}{\partial t'} = R^T\left[g - \frac{\partial u}{\partial t} - \varpi \times u - 2\,\Omega \times u\right]\ ,$$

that, taking into account 6.19, proves the first relation 6.17.

Moreover, by a straighforward calculation resulting from 6.16, it is easy to verify that the quantity 6.18 is invariant.■

6.2.4 How to manage holonomic constraints?

Thanks to Lagrange's multipliers, some modifications can be done to model other forces as the gravitation. For instance, let us consider Foucault's pendulum (Subsection 3.5.2). The bob moving on the sphere of centre $\boldsymbol{P}$, radius l, its position is subjected to the holonomic constraint:

$$f(r) = 1 - \frac{1}{l}\sqrt{x^2 + y^2 + (z-l)^2} = 0 \ ,$$

hence:

$$grad\, f = \frac{1}{l}\begin{pmatrix} -x \\ -y \\ l-z \end{pmatrix} \ .$$

The gravity g and the spinning Ω given by 3.78 being uniform, considering the gravitation potentials:

$$\phi = m \parallel g \parallel z, \qquad A = \Omega \times r \ ,$$

and Lagrange's multiplier S, the modified Euler-Lagrange equations 6.9 with $y = r$ allow to recover the generalized equations of motion 3.77 with the tension force along the thread:

$$F = S\, grad\, f \ .$$

where Lagrange's multiplier S is physically interpreted as the intensity of the tension force. Finally, we recover the equations 3.79, 3.80 and 3.81 of Foucault's pendulum motion by a variational formulation.

Part II

Galilean Dynamics and Thermodynamics of Continua

Chapter 7

Statics of 3D continua

7.1 Stresses

7.1.1 Stress tensor

The aim of this chapter is to study the static equilibrium of a bulky body occupying an open domain $\mathcal{V}$ of the 3D affine space (figure 7.1). First of all, we have to model the internal and external forces, defined in a global way at Subsection 2.3.3. We would like to give a more accurate description of them. To identified the internal forces, we isolate a part $\mathcal{V}_-$ of the body by cutting it along a smooth surface $\mathcal{S}$ of equation $f(r) = 0$ where f is a level-set function of class C^1 and conventionally negative in $\mathcal{V}_-$.

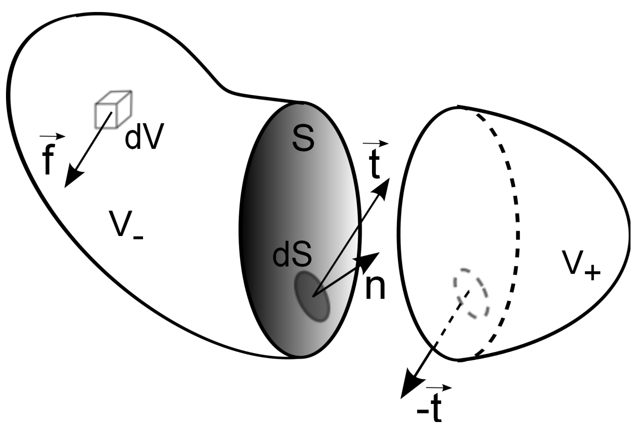

Figure 7.1: Stress vector

Drawing the free body diagram of the part $\mathcal{V}_-$, the distribution of internal forces acting upon it through the surface will be assumed at least continuous. At a given point $\boldsymbol{P}$ of interior of the surface, let us consider an infinitesimal **surface element** completely defined by its area $d\mathcal{S}$ and its orientation. Working in a Galilean coordinate system r, the surface element is perpandicular to the gradient of f of which the components are the partial derivative of f. In a Galilean coordinate system, the components of the unit normal to the surface, pointing away from the considered part, are:

$$n_i = \frac{1}{\|\frac{\partial f}{\partial r}\|} \frac{\partial f}{\partial r^i} \ .$$

which are the components of a covector $\mathbf{n}$, that is a 1-covariant tensor. According to the transformation law (2.10), we saw that the forces are vectors, that is 1-contravariant tensors. Let dF^j the components of the elementary internal force vector $\overrightarrow{\boldsymbol{dF}}_s$ acting at $\boldsymbol{P}$ upon $\mathcal{V}_-$ through the surface element $d\mathcal{S}$.

Definition 7.1 *The* ***stress vector*** *acting at the point* $\boldsymbol{P}$ *through the surface* $\mathcal{S}$ *(from* $\mathcal{V}_+$ *to* $\mathcal{V}_-$*) is*

$$\overrightarrow{\boldsymbol{t}} = \frac{\overrightarrow{\boldsymbol{dF}_s}}{d\mathcal{S}} \ ,$$

value of a map:

$$(\boldsymbol{P}, \boldsymbol{n}) \mapsto \overrightarrow{\boldsymbol{t}} = \boldsymbol{T}(\boldsymbol{P}, \boldsymbol{n}) \ .$$

In other words, if $\overrightarrow{\boldsymbol{\Delta F}}$ is the resultant of forces acting upon $\mathcal{V}_-$ through the small portion $\Delta\mathcal{S}$ of $\mathcal{S}$ around $\boldsymbol{P}$, it is the limit

$$\overrightarrow{\boldsymbol{t}} = \lim_{\Delta\mathcal{S}\to 0} \frac{\overrightarrow{\boldsymbol{\Delta F}}}{\Delta\mathcal{S}} \ .$$

By Newton's third law (2.5), the resultant of forces $\overrightarrow{\boldsymbol{\Delta F'}}$ acting upon $\mathcal{V}_+$ through $\Delta\mathcal{S}$ is such that:

$$\overrightarrow{\boldsymbol{\Delta F'}} = -\overrightarrow{\boldsymbol{\Delta F}} \ .$$

Dividing both members by the area and passing to the limit, the mutual stress vector of action and reaction are equal and opposite, that can read:

$$\boldsymbol{T}(\boldsymbol{P}, -\boldsymbol{n}) = -\boldsymbol{T}(\boldsymbol{P}, \boldsymbol{n}) \ . \tag{7.1}$$

In the same spirit, we consider the external forces can be modelised by the elementary force vector $\overrightarrow{\boldsymbol{dF}_v}$ acting at $\boldsymbol{P}$ upon the volume element $d\mathcal{V}$ around $\boldsymbol{P}$.

Definition 7.2 *The* ***volume force*** *acting at the point* $\boldsymbol{P}$ *upon the body* $\mathcal{V}$ *is*

$$\overrightarrow{\boldsymbol{f}} = \frac{\overrightarrow{\boldsymbol{dF}_v}}{d\mathcal{V}} \ .$$

Now we prove **Cauchy's tetrahedron theorem**:

Theorem 7.3 *If the maps* $(\boldsymbol{P}, \boldsymbol{n}) \mapsto \boldsymbol{T}(\boldsymbol{P}, \boldsymbol{n})$ *and* $\boldsymbol{P} \mapsto \overrightarrow{\boldsymbol{f}}(\boldsymbol{P})$ *are continuous, balance equation (2.12) implies that there exists a 2-contravariant tensor field* $\boldsymbol{P} \mapsto \boldsymbol{\sigma}(\boldsymbol{P})$*, called* ***Cauchy's stress tensor****, such that:*

$$\overrightarrow{\boldsymbol{t}} = \boldsymbol{\sigma}(\boldsymbol{P}) \cdot \boldsymbol{n} \ . \tag{7.2}$$

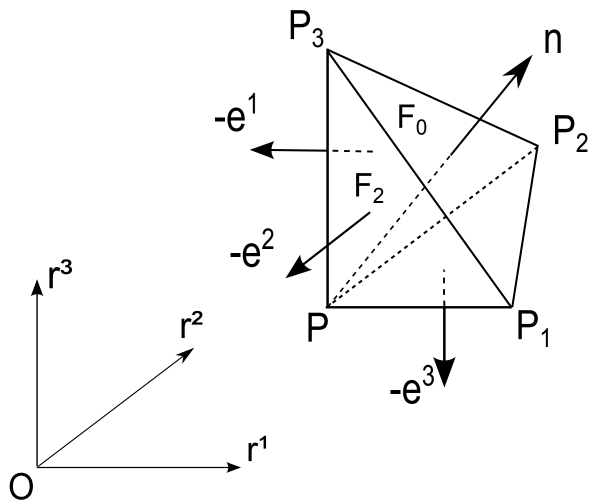

Figure 7.2: Stress vector

Proof. Because the set $\mathcal{V}$ is open, we can find, as a particular subdomain of $\mathcal{V}$, a thetrahedron $\mathcal{V}_t$ with vertex $\boldsymbol{P}$, three faces parallel to the coordinate planes and a face F_0 normal to $\boldsymbol{n}$ of area S_0 and at the distance h of $\boldsymbol{P}$. As indicated in figure 7.2, let F_i be the face opposite to another vertex $\boldsymbol{P}_j$, so its area is $\mathcal{S}_j = n_j \mathcal{S}_0$. The force balance equation (2.12) of the tetrahedron reads:

$$\sum_{k=0}^{3} \int_{F_k} \boldsymbol{T}(\boldsymbol{P}', \boldsymbol{n}) \, d\mathcal{S}(\boldsymbol{P}') + \int_{\mathcal{V}_t} \overrightarrow{\boldsymbol{f}}(\boldsymbol{P}') \, d\mathcal{V}(\boldsymbol{P}') = \vec{\boldsymbol{0}} \ .$$

and considering the projection on the axis $\boldsymbol{O}r^i$:

$$\sum_{k=0}^{3} \int_{F_k} T^i(\boldsymbol{P}', \boldsymbol{n}) \, d\mathcal{S}(\boldsymbol{P}') + \int_{\mathcal{V}_t} f^i(\boldsymbol{P}') \, d\mathcal{V}(\boldsymbol{P}') = 0 \ .$$

Taking into account the continuity hypothesis, we can apply the mean value theorem for integrals. Hence there exist points $\bar{\boldsymbol{P}}_k \in F_k$ and $\bar{\boldsymbol{P}} \in \mathcal{V}_t$ such that:

$$T^i(\bar{\boldsymbol{P}}_0, \boldsymbol{n}) \, \mathcal{S}_0 + \sum_{j=1}^{3} T^i(\bar{\boldsymbol{P}}_j, -\boldsymbol{e}^j) \, \mathcal{S}_j + f^i(\bar{\boldsymbol{P}}) \, \frac{h \, \mathcal{S}_0}{3} = 0 \ .$$

Thus, it holds:

$$T^i(\bar{\boldsymbol{P}}_0, \boldsymbol{n}) + \sum_{j=1}^{3} T^i(\bar{\boldsymbol{P}}_j, -\boldsymbol{e}^j)\, n_j + f^i(\bar{\boldsymbol{P}})\, \frac{h}{3} = 0 \ .$$

Keeping the covector $\boldsymbol{n}$ fixed when h approaches 0, the points $\boldsymbol{P}_i, \bar{\boldsymbol{P}}_j$ and $\bar{\boldsymbol{P}}$ coalesce into $\boldsymbol{P}$. Using again the continuity hypothesis, one has:

$$T^i(\boldsymbol{P}, \boldsymbol{n}) = -\sum_{j=1}^{3} T^i(\boldsymbol{P}, -\boldsymbol{e}^j)\, n_j \ .$$

The coordinate system being arbitrary, one has:

$$\boldsymbol{T}(\boldsymbol{P}, \boldsymbol{n}) = -\sum_{j=1}^{3} \boldsymbol{T}(\boldsymbol{P}, -\boldsymbol{e}^j)\, n_j \ . \tag{7.3}$$

Using next the continuity with respect to the second variable, let $\boldsymbol{n}$ approach a particular cobasis vector $\boldsymbol{e}^j$ in the previous relation, we obtain:

$$\boldsymbol{T}(\boldsymbol{P}, \boldsymbol{e}^j) = -\boldsymbol{T}(\boldsymbol{P}, -\boldsymbol{e}^j) \ ,$$

that is nothing else Newton's third law in the form (7.1). Taking into account of this, relation (7.3) leads to:

$$\boldsymbol{T}(\boldsymbol{P}, \sum_{j=1}^{3} n_j \boldsymbol{e}^j) = \boldsymbol{T}(\boldsymbol{P}, \boldsymbol{n}) = \sum_{j=1}^{3} n_j \boldsymbol{T}(\boldsymbol{P}, \boldsymbol{e}^j) \ .$$

that proves $\boldsymbol{T}$ is linear with respect to the second variable $\boldsymbol{n}$. As the value of $\boldsymbol{T}$ is a 1-contravariant tensor and $\boldsymbol{n}$ is a 1-covariant tensor, there exists a 2-contravariant tensor $\boldsymbol{\sigma}(\boldsymbol{P})$ such that $\boldsymbol{T}(\boldsymbol{P}, \boldsymbol{n})$ is obtained by contracting once the tensor $\boldsymbol{\sigma}(\boldsymbol{P}) \otimes \boldsymbol{n}$. ■

In a given basis $(\vec{\boldsymbol{e}}_i)$ and using the convention of summation, relation (7.2) reads:

$$t^i = \sigma^{ij} n_j \ .$$

In a new basis $(\vec{\boldsymbol{e}}'_s)$ obtain from the previous one through a transformation matrix $\breve{P}$, the new components are given by the transformation law of 2-contravariant tensors:

$$\sigma'^{st} = (\breve{P}^{-1})^s_i (\breve{P}^{-1})^t_j \sigma^{ij} \ , \tag{7.4}$$

which, according to (13.3), reads in matrix notations:

$$\sigma' = \breve{P}^{-1} \sigma\, \breve{P}^{-T} \ .$$

In particular, when working in Galilean coordinate systems, the transformation matrix is a an orthogonal transformation and we obtain the transformation law of Euclidean stress tensors:

$$\sigma' = R^T \sigma\, R\ . \tag{7.5}$$

To model realistic situations, is is often useful to consider piecewize continuous distributions of forces. Let two regions $\mathcal{V}_-$ and $\mathcal{V}_+$ where the stress field is continuous and separated by a discontinuity surface $\mathcal{S}$. According to Newton's third law in the form (7.1), the only continuity requirement crossing $\mathcal{S}$ is the continuity of the stress vector:

$$\left[\overrightarrow{\boldsymbol{t}}\right] = \vec{0}\ .$$

7.1.2 Local equilibrium equations

Now we prove:

Theorem 7.4 *If the maps $r \mapsto \sigma(r)$ is continuously differentiable, balance equation (2.12) implies the* ***local equilibrium equations of 3D continua****:*

$$\boxed{(div\,\sigma)^T + f = 0} \tag{7.6}$$

$$\boxed{\sigma^T = \sigma} \tag{7.7}$$

Proof. Working in a given Galilean coordinate system, let dF be the 3-column gathering the components dF^i of the elementary force $\overrightarrow{\boldsymbol{dF}}$ acting at $\boldsymbol{P}$ of coordinates r^i. Its torsor (2.11) about the origin $\boldsymbol{O}$ is, owing to (12.9):

$$d\breve{\mu} = \begin{pmatrix} 0 & dF^T \\ -dF & -j(r \times dF) \end{pmatrix} = \begin{pmatrix} 0 & dF^T \\ -dF & r\, dF^T - dF\, r^T \end{pmatrix}\ .$$

For a arbitrary subdomain $\mathcal{V}$ of the considered body, the resultant torsor is the integral of the elementary torsor of every infinitesimal parts corresponding to forces $\overrightarrow{\boldsymbol{dF}}_v$ on $\mathcal{V}$ and $\overrightarrow{\boldsymbol{dF}}_s$ on its boundary $\partial\mathcal{V}$:

$$\breve{\mu}(\mathcal{V}) = \int_{\partial\mathcal{V}} d\breve{\mu}_s(r') + \int_{\mathcal{V}} d\breve{\mu}_v(r') = \begin{pmatrix} 0 & F_{\mathcal{V}}^T \\ -F_{\mathcal{V}} & J_{\mathcal{V}} \end{pmatrix}\ ,$$

where:

$$F_{\mathcal{V}} = \int_{\partial\mathcal{V}} dF_s(r') + \int_{\mathcal{V}} dF_v(r') = \int_{\partial\mathcal{V}} t(r')\, d\mathcal{S}(r') + \int_{\mathcal{V}} f(r')\, d\mathcal{V}(r')\ ,$$

$$J_{\mathcal{V}} = \int_{\partial\mathcal{V}} (r\,(t(r'))^T - t(r')\,r^T)\,d\mathcal{S}(r') + \int_{\mathcal{V}} (r\,(f(r'))^T - f(r')\,r^T)\,d\mathcal{V}(r')\ .$$

Working now with tensor components and simplified notations, taking into account Theorem 7.3, the force resultant is represented by:

$$F^i_{\mathcal{V}} = \int_{\partial\mathcal{V}} t^i\,d\mathcal{S} + \int_{\mathcal{V}} f^i\,d\mathcal{V} = \int_{\partial\mathcal{V}} \sigma^{ij} n_j\,d\mathcal{S} + \int_{\mathcal{V}} f^i\,d\mathcal{V}\ .$$

Transforming the first term by the divergence theorem for tensor fields and owing to the balance equation, we have:

$$F^i_{\mathcal{V}} = \int_{\mathcal{V}} \left(\frac{\partial\sigma^{ij}}{\partial r^j} + f^i \right) d\mathcal{V} = 0\ ,$$

for every subdomain $\mathcal{V}$, therefore one has at any point r:

$$\frac{\partial\sigma^{ij}}{\partial r^j} + f^i = 0\ , \qquad (7.8)$$

As the choice of the Galilean coordinate system is arbitrary, this proves the force equilibrium equation (7.6). In a similar way, the moment resultant matrix $J_{\mathcal{V}}$ is represented by:

$$J^{ik}_{\mathcal{V}} = \int_{\partial\mathcal{V}} (r^i t^k - t^i r^k)\,d\mathcal{S} + \int_{\mathcal{V}} (r^i f^k - f^i r^k)\,d\mathcal{V}\ .$$

Transforming the first integral by the divergence theorem, one has:

$$\int_{\partial\mathcal{V}} (r^i t^j - t^i r^j)\,d\mathcal{S} = \int_{\partial\mathcal{V}} (r^i \sigma^{kj} - r^k \sigma^{ij}) n_j\,d\mathcal{S} = \int_{\mathcal{V}} \frac{\partial}{\partial r^j} (r^i \sigma^{kj} - r^k \sigma^{ij})\,d\mathcal{V}\ ,$$

$$\int_{\partial\mathcal{V}} (r^i t^j - t^i r^j)\,d\mathcal{S} = \int_{\mathcal{V}} \left(\delta^i_j \sigma^{kj} - \delta^k_j \sigma^{ij} + r^i \frac{\partial\sigma^{kj}}{\partial r^j} - r^k \frac{\partial\sigma^{ij}}{\partial r^j} \right) d\mathcal{V}\ .$$

Taking into account the force equilibrium (7.8) leads to:

$$J^{ik}_{\mathcal{V}} = \int_{\mathcal{V}} \left(\sigma^{ki} - \sigma^{ik} \right) d\mathcal{V}\ ,$$

for every subdomain $\mathcal{V}$, therefore one has at any point r:

$$\sigma^{ki} = \sigma^{ik}\ . \qquad (7.9)$$

As the choice of the Galilean coordinate system is arbitrary, this proves the moment equilibrium equation (7.7). ■

7.2 Torsors

7.2.1 Continuum torsor

Thanks to Cauchy's tetrahedron theorem 7.3, we built a linear tensor, the stress tensor. At the previous Section, we used the resultant torsor of the body but without discussing its tensor status. In fact, the torsors are affine tensors. For instance, the torsor of a force $\vec{\boldsymbol{F}}$ acting at point $\boldsymbol{P}$ is the skew-symmetric 2-contravariant affine tensor:

$$\breve{\boldsymbol{\mu}} = \boldsymbol{P} \otimes \vec{\boldsymbol{F}} - \vec{\boldsymbol{F}} \otimes \boldsymbol{P} \ , \tag{7.10}$$

but, up to now, we was working only with its local representation in a given affine frame f where, if r is the position of $\boldsymbol{P}$ and $\vec{\boldsymbol{F}}$ is represented by the column F, the torsor $\boldsymbol{\mu}$ is represented by the 4×4 skew-symmetric matrix:

$$\breve{\mu} = \begin{pmatrix} 1 \\ r \end{pmatrix} \begin{pmatrix} 0 \\ F \end{pmatrix}^T - \begin{pmatrix} 0 \\ F \end{pmatrix} \begin{pmatrix} 1 \\ r \end{pmatrix}^T \ ,$$

hence, using (12.9), we recover (2.4) with $M = r \times F$ while the transformation law (2.5) of torsors is nothing else the one (13.9) of the 2-contravariant affine tensors. Also, taking into account the decomposition in the affine frame of the point $\boldsymbol{P}$ and the force vector:

$$\boldsymbol{P} = \boldsymbol{O} + r^k \vec{\boldsymbol{e}}_k, \qquad \vec{\boldsymbol{F}} = F^i \vec{\boldsymbol{e}}_i \ ,$$

the relation (7.10) leads to the decomposition of the force torsor:

$$\breve{\boldsymbol{\mu}} = F^i \left(\boldsymbol{O} \otimes \vec{\boldsymbol{e}}_i - \vec{\boldsymbol{e}}_i \otimes \boldsymbol{O} \right) + \left(r^k F^i - r^i F^k \right) \vec{\boldsymbol{e}}_k \otimes \vec{\boldsymbol{e}}_i \ .$$

Similarly, let us construct the torsor of the stress vector:

$$\frac{d\breve{\boldsymbol{\mu}}_s}{d\mathcal{S}} \left(\boldsymbol{P}, \boldsymbol{n} \right) = \boldsymbol{P} \otimes \vec{\boldsymbol{t}}(\boldsymbol{P}, \boldsymbol{n}) - \vec{\boldsymbol{t}}(\boldsymbol{P}, \boldsymbol{n}) \otimes \boldsymbol{P} \ .$$

Clearly, it is linearly depending on the covector $\boldsymbol{n}$, that leads to consider a vector valued torsor $\breve{\boldsymbol{\mu}}_{\boldsymbol{\sigma}}$, called the **stress torsor**, such that:

$$\boldsymbol{n}(\breve{\boldsymbol{\mu}}_{\boldsymbol{\sigma}}(\boldsymbol{P})) = \frac{d\breve{\boldsymbol{\mu}}_s}{d\mathcal{S}} \left(\boldsymbol{P}, \boldsymbol{n} \right) \ .$$

Hence,

$$\boldsymbol{n}(\breve{\boldsymbol{\mu}}_{\boldsymbol{\sigma}}) = \boldsymbol{P} \otimes (\boldsymbol{\sigma} \cdot \boldsymbol{n}) - (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \otimes \boldsymbol{P} \ . \tag{7.11}$$

On this ground, it is worth to generalize the torsors in the following way.

Definition 7.5 *A* ***continuum torsor*** *is a skew-symmetric* 2*-contravariant affine tensor* $\boldsymbol{\mu}$ *with vector value:*

$$\boldsymbol{\mu}(\boldsymbol{\Psi}, \bar{\boldsymbol{\Psi}}) = -\boldsymbol{\mu}(\bar{\boldsymbol{\Psi}}, \boldsymbol{\Psi}) \ .$$

For what we are concerned now, the continuum is 3D and we denote the torsors $\breve{\boldsymbol{\mu}}$. In the affine frame $(\boldsymbol{O}, (\vec{\boldsymbol{e}}_i))$, its value is:

$$\vec{\boldsymbol{U}} = \breve{\boldsymbol{\mu}}(\boldsymbol{\Psi}, \bar{\boldsymbol{\Psi}}) = \breve{\mu}^j((\chi, \Phi), (\bar{\chi}, \bar{\Phi}))\, \vec{\boldsymbol{e}}_j \ ,$$

$$\vec{\boldsymbol{U}} = \left[T^{ij}(\chi \bar{\Phi}_i - \bar{\chi}\Phi_i) + J^{kij}\, \Phi_k \bar{\Phi}_i \right] \vec{\boldsymbol{e}}_j \ ,$$

where T^{ij} and $J^{kij} = -J^{ikj}$ are the components of the torsor. Taking into account $\chi = \hat{\boldsymbol{O}}(\boldsymbol{\Psi})$ and $\Phi_i = \hat{\boldsymbol{e}}_i(\boldsymbol{\Psi})$, it reads:

$$\breve{\boldsymbol{\mu}} = \breve{\boldsymbol{\mu}}^j \vec{\boldsymbol{e}}_j, \qquad \breve{\boldsymbol{\mu}}^j = T^{ij}\,(\boldsymbol{O} \otimes \vec{\boldsymbol{e}}_i - \vec{\boldsymbol{e}}_i \otimes \boldsymbol{O}) + J^{kij}\, \vec{\boldsymbol{e}}_k \otimes \vec{\boldsymbol{e}}_i \ .$$

Let $(\boldsymbol{O}', (\vec{\boldsymbol{e}}'_i))$ be a new affine frame obtained from the old one through an affine transformation $\breve{a} = (k, \breve{P})$. Hence, the transformation law of the torsor is:

$$T'^{st} = (\breve{P}^{-1})^s_i (\breve{P}^{-1})^t_j T^{ij} \ , \tag{7.12}$$

$$J'^{rst} = \left[(\breve{P}^{-1})^r_l (\breve{P}^{-1})^s_i J^{lij} + k'^r \left\{ (\breve{P}^{-1})^s_i T^{ij} \right\} - \left\{ (\breve{P}^{-1})^r_i T^{ij} \right\} k'^s \right] (\breve{P}^{-1})^t_j \ . \tag{7.13}$$

with: $k' = -\breve{P}^{-1} k$.

Now, let us come back to the stress torsor. Taking into account the decomposition in the affine frame of the point $\boldsymbol{P}$ and the stress vector:

$$\boldsymbol{P} = \boldsymbol{O} + r^k \vec{\boldsymbol{e}}_k, \qquad \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{\sigma}^{ij} \vec{\boldsymbol{e}}_i n_j \ ,$$

the relation (7.11) leads to the decomposition of the stress torsor:

$$\breve{\boldsymbol{\mu}}_{\boldsymbol{\sigma}} = \breve{\boldsymbol{\mu}}^j_{\boldsymbol{\sigma}} \vec{\boldsymbol{e}}_j, \qquad \breve{\boldsymbol{\mu}}^j_{\boldsymbol{\sigma}} = \sigma^{ij}\,(\boldsymbol{O} \otimes \vec{\boldsymbol{e}}_i - \vec{\boldsymbol{e}}_i \otimes \boldsymbol{O}) + (r^k \sigma^{ij} - r^i \sigma^{kj})\, \vec{\boldsymbol{e}}_k \otimes \vec{\boldsymbol{e}}_i \ .$$

hence, the components of the stress tensor are:

$$T^{ij} = \sigma^{ij}, \qquad J^{kij} = r^k \sigma^{ij} - r^i \sigma^{kj} \ . \tag{7.14}$$

For the stress tensor, it is worth observing that (7.12) is nothing else the transformation law (7.4) of the stress tensors. Also, the J^{lij} components of $\breve{\boldsymbol{\mu}}_{\boldsymbol{\sigma}}$ in the affine frame $(\boldsymbol{P}, (\vec{\boldsymbol{e}}_i))$ vanish because $r = 0$ if $\boldsymbol{P}$ is the origin. In fact, this condition caracterizes a stress torsor, according to the next theorem.

Theorem 7.6 *A continuum torsor $\breve{\boldsymbol{\mu}}$ is a stress torsor if and only if the components J^{lij} of $\breve{\boldsymbol{\mu}}(\boldsymbol{P})$ in the affine frame $(\boldsymbol{P}, (\vec{\boldsymbol{e}}_i))$ vanish.*

Proof. We just showed the condition is necessary. To proove it is also sufficient, let us suppose $J^{lij} = 0$ for the representation of $\breve{\boldsymbol{\mu}}(\boldsymbol{P})$ in $(\boldsymbol{P}, (\vec{\boldsymbol{e}}_i))$. According to the transformation laws (7.12) and (7.13), the components of the continuum torsor $\breve{\boldsymbol{\mu}}(\boldsymbol{P})$ in $(\boldsymbol{O}, (\vec{\boldsymbol{e}}_i))$ are deduced from the ones in $(\boldsymbol{P}, (\vec{\boldsymbol{e}}_i))$ through a translation $k = S^{-1}(\overrightarrow{\boldsymbol{OP}}) = r$ (thus the transformation matrix P is the identity):

$$T'^{ij} = T^{ij}, \qquad J'^{lij} = r^l T^{ij} - r^i T^{lj} \ .$$

We can put:

$$\breve{\boldsymbol{\mu}} = \breve{\boldsymbol{\mu}}^j \vec{\boldsymbol{e}}_j, \qquad \breve{\boldsymbol{\mu}}^j = T^{ij}\,(\boldsymbol{O} \otimes \vec{\boldsymbol{e}}_i - \vec{\boldsymbol{e}}_i \otimes \boldsymbol{O}) + (r^l T^{ij} - r^i T^{lj})\,\vec{\boldsymbol{e}}_l \otimes \vec{\boldsymbol{e}}_i \ .$$

By straightforward calculations, we obtain:

$$\boldsymbol{n}(\breve{\boldsymbol{\mu}}) = \boldsymbol{P} \otimes (\boldsymbol{T} \cdot \boldsymbol{n}) - (\boldsymbol{T} \cdot \boldsymbol{n}) \otimes \boldsymbol{P} = \boldsymbol{n}(\breve{\boldsymbol{\mu}}_{\boldsymbol{T}}) \ .$$

As the linear form $\boldsymbol{n}$ is arbitrary, the continuum torsor $\breve{\boldsymbol{\mu}}$ is the stress torsor $\breve{\boldsymbol{\mu}}_{\boldsymbol{T}}$. ■

7.2.2 Cauchy's continuum

At first glance, the local equilibrium equations (7.8) and (7.9) are rather dissimilar. Considering the concept of torsor, is it possible to reveal an underlying structure? Considering the components of a continuum torsor $\breve{\boldsymbol{\mu}}$ in the affine frame $(\boldsymbol{O}, (\vec{\boldsymbol{e}}_i))$, we claim that its **divergence** is the scalar valued torsor:

$$\tilde{\boldsymbol{div}}\, \breve{\boldsymbol{\mu}} = \frac{\partial T^{ij}}{\partial r^j}\,(\boldsymbol{O} \otimes \vec{\boldsymbol{e}}_i - \vec{\boldsymbol{e}}_i \otimes \boldsymbol{O}) + \frac{\partial J^{kij}}{\partial r^j}\,\vec{\boldsymbol{e}}_k \otimes \vec{\boldsymbol{e}}_i \ . \tag{7.15}$$

Also, let us define the torsor of the volume force:

$$\boldsymbol{\mu}_{\vec{\boldsymbol{f}}} = \frac{d\breve{\boldsymbol{\mu}}_v}{d\mathcal{S}}\,(\boldsymbol{P}) = \boldsymbol{P} \otimes \vec{\boldsymbol{f}}(\boldsymbol{P}) - \vec{\boldsymbol{f}}(\boldsymbol{P}) \otimes \boldsymbol{P} \ ,$$

$$\breve{\boldsymbol{\mu}}_{\vec{\boldsymbol{f}}} = f^i\,(\boldsymbol{O} \otimes \vec{\boldsymbol{e}}_i - \vec{\boldsymbol{e}}_i \otimes \boldsymbol{O}) + (r^k f^i - r^i f^k)\,\vec{\boldsymbol{e}}_k \otimes \vec{\boldsymbol{e}}_i \ . \tag{7.16}$$

Thus the equation:

$$\boxed{\tilde{\boldsymbol{div}}\, \breve{\boldsymbol{\mu}}_{\boldsymbol{\sigma}} + \breve{\boldsymbol{\mu}}_{\vec{\boldsymbol{f}}} = \boldsymbol{0}} \tag{7.17}$$

allows to recover (7.8) and:

$$r^k \left(\frac{\partial \sigma^{ij}}{\partial r^j} + f^i \right) - r^i \left(\frac{\partial \sigma^{kj}}{\partial r^j} + f^k \right) + \sigma^{ik} - \sigma^{ki} = 0 \ ,$$

which, owing to (7.8), is nothing else (7.9).

The drawback of definition (7.15) of the torsor divergence is that it is not general with respect to its status of affine tensor. In the spirit of Chapter 4, we introduce a **covariant divergence** of a continuum torsor $\breve{\boldsymbol{\mu}}$, considering the components of $\breve{\boldsymbol{\mu}}(\boldsymbol{P})$ in the current affine frame $(\boldsymbol{P}, (\vec{\boldsymbol{e}}_i))$ (and not in $(\boldsymbol{O}, (\vec{\boldsymbol{e}}_i))$!). We work in steps:

- we calculate the divergence at $\boldsymbol{P}'$ of the component system $\breve{\mu}$ of $\breve{\boldsymbol{\mu}}\,(\boldsymbol{P}')$ in the affine frame $(\boldsymbol{P}, (\vec{\boldsymbol{e}}_i))$ where $\boldsymbol{P}$ is a neighbour point of $\boldsymbol{P}'$,
- we consider its limit as $\boldsymbol{P}'$ approaches $\boldsymbol{P}$.

Firstly, we express $\breve{\mu}$ with respect to the component system $\breve{\mu}'$ of $\breve{\boldsymbol{\mu}}\,(\boldsymbol{P}')$ in the affine frame $(\boldsymbol{P}', (\vec{\boldsymbol{e}}_i))$ by considering a translation $k' = r' - r$ (hence the transformation matrix P is the identity). Transformation laws (7.12) and (7.13) give:

$$T^{ij} = T'^{ij}, \qquad J^{lij} = J'^{lij} + (r^l - r'^l)T^{ij} - (r^i - r'^i)T^{lj} \ .$$

Hence the components of the divergence of $\breve{\boldsymbol{\mu}}$ are:

$$\frac{\partial T^{ij}}{\partial r'^j} = \frac{\partial T'^{ij}}{\partial r'^j}, \qquad \frac{\partial J^{kij}}{\partial r'^j} = \frac{\partial J'^{kij}}{\partial r'^j} + T^{li} - T^{il} \ .$$

Considering their limits as $\boldsymbol{P}'$ approaches $\boldsymbol{P}$:

$$\tilde{\nabla}_j T^{ij} = \lim_{\boldsymbol{P}' \to \boldsymbol{P}} \frac{\partial T^{ij}}{\partial r'^j}, \qquad \tilde{\nabla}_j J^{lij} = \lim_{\boldsymbol{P}' \to \boldsymbol{P}} \frac{\partial J^{lij}}{\partial r'^j} \ ,$$

since r' approaches r, T'^{ij} approaches T^{ij} and J'^{lij} approaches J^{lij}, it holds:

$$\tilde{\nabla}_j T^{ij} = \frac{\partial T^{ij}}{\partial r^j}, \qquad \tilde{\nabla}_j J^{lij} = \frac{\partial J^{lij}}{\partial r^j} + T^{li} - T^{il} \ . \tag{7.18}$$

Then the **covariant affine divergence** of the torsor field $\breve{\boldsymbol{\mu}}$ is:

$$\tilde{\boldsymbol{div}}\, \breve{\boldsymbol{\mu}} = \tilde{\nabla}_j T^{ij}\,(\boldsymbol{P} \otimes \vec{\boldsymbol{e}}_i - \vec{\boldsymbol{e}}_i \otimes \boldsymbol{P}) + \tilde{\nabla}_j J^{lij}\,\vec{\boldsymbol{e}}_l \otimes \vec{\boldsymbol{e}}_i \ . \tag{7.19}$$

With this new definition, we can verify that the stress torsor and the volume force torsor verify the local equilibrium equations in the form (7.17). Indeed, owing to (9.29), we have in the current affine frame $(\boldsymbol{P}, (\vec{\boldsymbol{e}}_i))$:

$$\breve{\boldsymbol{\mu}}_{\vec{\boldsymbol{f}}} = f^i\,(\boldsymbol{P} \otimes \vec{\boldsymbol{e}}_i - \vec{\boldsymbol{e}}_i \otimes \boldsymbol{P}) \ .$$

Besides, according to Theorem 7.6:

$$\tilde{\boldsymbol{div}}\, \breve{\boldsymbol{\mu}}_{\boldsymbol{\sigma}} = \frac{\partial \sigma^{ij}}{\partial r^j}\,(\boldsymbol{P} \otimes \vec{\boldsymbol{e}}_i - \vec{\boldsymbol{e}}_i \otimes \boldsymbol{P}) + (\sigma^{li} - \sigma^{il})\,\vec{\boldsymbol{e}}_l \otimes \vec{\boldsymbol{e}}_i \ .$$

In short, we introduce a general definition of a continuum torsor and its divergence as affine tensor. Particularizing it to the stress torsors, we recover the local equilibrium equation (7.6) and (7.7) from the more compact Formula (7.17).

Definition 7.7 *Continua of which the torsor is a stress torsor are called* ***Cauchy's continua****.*

As consequence of equilibrium equations, the stress tensors of Cauchy's media are symmetric.

7.3 Invariants of the stress tensor

As the stress tensor $\boldsymbol{\sigma}$ is represented in an orthonormal basis by a symmetric matrix σ, it is diagonalizable and its eigenvalues $\sigma_1, \sigma_2, \sigma_3$ are real numbers called **principal stresses**. They are obtained by solving the characteristic equation:

$$\det(\sigma - \lambda\, 1_{\mathbb{R}^3}) = 0\ .$$

Owing to the transformation law (7.5) of Euclidian stress tensors, this equation is invariant under any orthogonal transformation R:

$$\det(\sigma' - \lambda\, 1_{\mathbb{R}^3}) = \det(R^T \sigma R - \lambda\, R^T 1_{\mathbb{R}^3} R) = \det\left[R^T(\sigma - \lambda\, 1_{\mathbb{R}^3})R\right]\ ,$$

$$\det(\sigma' - \lambda\, 1_{\mathbb{R}^3}) = (det(R))^2 \det(\sigma - \lambda\, 1_{\mathbb{R}^3}) = \det(\sigma - \lambda\, 1_{\mathbb{R}^3})\ .$$

As σ is a 3×3 matrix, the characteristic equation reads:

$$\det(\sigma - \lambda\, 1_{\mathbb{R}^3}) = -\lambda^3 + \iota_1(\sigma)\lambda^2 - \iota_2(\varepsilon)\lambda + \iota_3(\varepsilon) = 0\ ,$$

where:

$$\iota_1(\sigma) = Tr(\sigma) = \sigma_1 + \sigma_2 + \sigma_3\ ,$$

$$\iota_2(\sigma) = \frac{1}{2}\left[(Tr(\sigma))^2 - Tr(\sigma^2)\right] = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\ ,$$

$$\iota_3(\sigma) = \det(\sigma) = \sigma_1\sigma_2\sigma_3\ .$$

are called the **principal invariants** of σ. We denote:

$$\iota(\sigma) = (\iota_1(\sigma), \iota_2(\sigma), \iota_3(\sigma))\ .$$

As any function of invariants is also invariant, they allow to generate other systems of invariants, for instance the one of the principal stresses.

Chapter 8

Elasticity and elementary theory of beams

8.1 Strains

In Subsection 4.3.1, we consider the currently met situations where the deformations of materials are small. For a bulky body $\mathcal{V}$, we would like generalize the Definition (4.25) of the extension of trusses. A particle of the body, initially at $\boldsymbol{P}$, has position $\boldsymbol{P'}$ when the body is subjected to given external forces. We claim that the difference $\overrightarrow{\boldsymbol{PP'}}$, called **displacement** is a smooth vector field $\boldsymbol{P} \mapsto \overrightarrow{\boldsymbol{u}}(\boldsymbol{P})$.

A body which does not undergoes deformations is a rigid body (Definition 3.5). Its motion preserves material length and angles or, in other words, the metric tensor $\breve{\boldsymbol{G}}$ of which Gram's matrix $\breve{G}$ is the identity in any Galilean coordinate system. To characterize the small deformation, we hope to build a tensor field depending on the displacement field and vanishing for every rigid motion. This suggests to study how the covariant metric tensor is perturbated by a rigid motion. We are working in two steps:

- Construct the curve $t \mapsto \boldsymbol{P'} = \varphi_t(\boldsymbol{P})$ solution of the ordinary differential equation
$$\frac{d}{dt}(\boldsymbol{\varphi}_t(\boldsymbol{P})) = \overrightarrow{\boldsymbol{u}}(\boldsymbol{\varphi}_t(\boldsymbol{P})) \ , \tag{8.1}$$
with the initial condition $\boldsymbol{\varphi}_0(\boldsymbol{P}) = \boldsymbol{P}$.

- Consider the pull-back of the metric tensor at $\boldsymbol{P'} = \boldsymbol{\varphi}_t(\boldsymbol{P})$ and compare it to the metric at $\boldsymbol{P}$, next divide by t and calculate half the limit when t approaches zero:
$$\boldsymbol{\varepsilon} = \frac{1}{2}\, lim_{t\to 0}\frac{1}{t}\left[\boldsymbol{\varphi}_t^{*}\breve{\boldsymbol{G}} - \breve{\boldsymbol{G}}\right] \ . \tag{8.2}$$

As the metric tensor, this quantity is a symmetric 2-covariant tensor called the **strain tensor** (but it is not a metric!). Now, let us determine it explicitly with respect to the displacement field. The question being local, we can work with a Galilean coordinate system in which the solution of (8.1) is expanded as:

$$r' = \varphi_t(r) = r + t\, u(r) + O(t^2) \ ,$$

where u, r and r' are the columns gathering respectively the contravariant components of the displacement $\vec{\boldsymbol{u}}$, the coordinates of $\boldsymbol{P}$ and $\boldsymbol{P}' = \varphi_t(\boldsymbol{P})$. By differentiation, the tangent map to $\boldsymbol{\varphi}_t$ is represented by the jacobean matrix:

$$\frac{\partial r'}{\partial r} = 1_{\mathbb{R}^3} + t\, \frac{\partial u}{\partial r} + O(t^2) \ .$$

The strain tensor $\boldsymbol{\varepsilon}$ is represented by the symmetric 3×3 matrix :

$$\varepsilon = \frac{1}{2}\, lim_{t \to 0} \frac{1}{t} \left[\left(\frac{\partial r'}{\partial r} \right)^T \breve{G}'\, \frac{\partial r'}{\partial r} - \breve{G} \right] \ ,$$

where $\breve{G}'$ is Gram's matrix at r'. Hence, one as:

$$\varepsilon = \frac{1}{2}\, lim_{t \to 0} \frac{1}{t} \left[\breve{G}' + t\, \breve{G}'\, \frac{\partial u}{\partial r} + t\, \left(\frac{\partial u}{\partial r} \right)^T \breve{G}' - \breve{G} + O(t^2) \right] \ .$$

In a Galilean coordinate system, Gram's matrix is constant, hence:

$$\varepsilon = \frac{1}{2} \left[\breve{G}\, \frac{\partial u}{\partial r} + \left(\frac{\partial u}{\partial r} \right)^T \breve{G} \right] \ .$$

Going back to indicial notations gives:

$$\varepsilon_{ij} = \frac{1}{2} \left(\breve{G}_{ik}\, \frac{\partial u^k}{\partial r^j} + \frac{\partial u^k}{\partial r^i} \breve{G}_{kj} \right) = \frac{1}{2} \left(\frac{\partial u_i}{\partial r^j} + \frac{\partial u_j}{\partial r^i} \right) \ .$$

As the contravariant and covariant components of the displacement are identical, this relation read in matrix notation:

$$\varepsilon = \frac{1}{2} \left[\frac{\partial u}{\partial r} + \left(\frac{\partial u}{\partial r} \right)^T \right] = grad_s\, u \ .$$

We have to check this quantity vanishes for every rigid motion. Indeed, such a motion (3.26) is compound of a translation and a rotation. Under the hypothesis of small deformations, the rotation can be considered as infinitesimal hence of the form (3.24), that leads to the following modeling of the small **rigid displacement** fields:

$$u(r) = j(d\psi)\, r + dc = d\psi \times r + dc \ ,$$

where $d\psi, dc \in \mathbb{R}^3$ are independent of r. For such a field, the derivative is skew-symmetric:

$$\frac{\partial u}{\partial r} = j(d\psi) \ .$$

hence its symmetric gradient vanishes. In other words, the condition of vanishing strain tensor is necessary for the motion to be rigid. It is also sufficient as **Kirchhoff's theorem** claims it:

Theorem 8.1 *If the domain $\mathcal{V}$ occupied by the body is connected, the displacement field is rigid if and only if the strain field vanishes.*

Proof. We just showed the condition is necessary. To proove it is also sufficient, let us assume that the strain is null, then the derivative of the displacement is skew-symmetric. There exists a field $r \mapsto \omega(r) \in \mathbb{R}^3$ such that:

$$\omega = j^{-1}\left(\frac{\partial u}{\partial r}\right) \ .$$

Let us prove it is independent of r. Indeed, one has:

$$\frac{1}{2}\, curl\, u = \frac{1}{2}\, j^{-1}\left(\frac{\partial u}{\partial r} - \left(\frac{\partial u}{\partial r}\right)^T\right) = \omega \ ,$$

hence:

$$div\, \omega = \frac{1}{2}\, div\, (curl\, u) = 0 \ .$$

On the other hand, one has:

$$curl\, (j(\omega)) = -curl\, ((j(\omega))^T) = -curl\, (grad\, u) = 0$$

thus:

$$\frac{\partial \omega}{\partial r} = curl\, (j(\omega)) + div\, \omega \cdot 1_{\mathbb{R}^3} = 0 \ .$$

The field is uniform. The body being connected, we obtain by integration:

$$u(r) = j(\omega)\,(r - r_0) + u(r_0) \ ,$$

that achieves the proof. ■

The transformation law of 2-contravariant tensors gives:

$$\varepsilon'_{st} = (\breve{P}^T)^i_s \breve{P}^j_t\, \varepsilon_{ij} \ , \tag{8.3}$$

which, according to (13.2), reads in matrix notations:

$$\varepsilon' = \breve{P}^T \varepsilon\, \breve{P} \ .$$

In particular, when working in Galilean coordinate systems, the transformation matrix is a an orthogonal transformation and we obtain the transformation law of the Euclidean strain tensors:

$$\varepsilon' = R^T \varepsilon\, R \ . \tag{8.4}$$

As the strain tensor $\boldsymbol{\varepsilon}$ is represented in an orthonormal basis by a symmetric matrix ε, it is diagonalizable and its eigenvalues $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are real numbers called **principal strains**. They can be deduced from the corresponding system $\iota(\varepsilon)$ of principal invariants.

8.2 Internal work and power

Let us consider an arbitrary subdomain $\mathcal{V}$ of the body. According to Definition 7.1, the elementary surface force acting at a point $\boldsymbol{P}$ of the boundary $\partial\mathcal{V}$ upon $\mathcal{V}$ through $d\mathcal{S}$ is:

$$\overrightarrow{\boldsymbol{dF}}_s = \overrightarrow{\boldsymbol{t}}\, d\mathcal{S} \ .$$

The elementary work provided by the force to produce a displacement $\overrightarrow{\boldsymbol{u}}$ of point $\boldsymbol{P}$ is:

$$d\mathcal{W}_s = \overrightarrow{\boldsymbol{dF}}_s \cdot \overrightarrow{\boldsymbol{u}} = \overrightarrow{\boldsymbol{t}} \cdot \overrightarrow{\boldsymbol{u}}\, d\mathcal{S} \ .$$

Using matrix notations and taking into account (7.2) and (7.7), one has:

$$d\mathcal{W}_s = t^T u\, d\mathcal{S} = (\sigma\, n)^T u\, d\mathcal{S} = n^T \sigma\, u\, d\mathcal{S} \ .$$

According to Definition 7.2, the elementary volume force acting at a point $\boldsymbol{P}$ of $\mathcal{V}$ upon the volume element $d\mathcal{V}$ around $\boldsymbol{P}$ is:

$$\overrightarrow{\boldsymbol{dF}}_v = \overrightarrow{\boldsymbol{f}}\, d\mathcal{V} \ .$$

The elementary work provided by the force to produce a displacement $\overrightarrow{\boldsymbol{u}}$ of point $\boldsymbol{P}$ is:

$$d\mathcal{W}_v = \overrightarrow{\boldsymbol{dF}}_v \cdot \overrightarrow{\boldsymbol{u}} = \overrightarrow{\boldsymbol{f}} \cdot \overrightarrow{\boldsymbol{u}}\, d\mathcal{V} = f^T u\, d\mathcal{V} \ .$$

The total work provided by external forces is:

$$\mathcal{W} = \int_{\partial\mathcal{V}} n^T \sigma\, u\, d\mathcal{S} + \int_{\mathcal{V}} f^T u\, d\mathcal{V} \ .$$

Owing to the internal equilibrium equations (7.6) and Green formula (13.17), one has:

$$\mathcal{W} = \int_{\mathcal{V}} \left(div\,(\sigma\, u) - (div\,\sigma)\, u\right)\, d\mathcal{V} \ .$$

Taking into account (13.12) and once again the symmetry (7.7) of the stress tensor, it holds:

$$\mathcal{W} = \int_{\mathcal{V}} Tr\, \left(\sigma\, \frac{\partial u}{\partial r}\right)\, d\mathcal{V} = \int_{\mathcal{V}} Tr\, \left(\sigma\, \varepsilon\right)\, d\mathcal{V} \ .$$

Taking into account the continuity hypothesis, we can apply the mean value theorem for integrals. Hence there exists a points $\bar{\boldsymbol{P}} \in \mathcal{V}$ such that:

$$\mathcal{W} = Tr\, \left(\sigma(\bar{\boldsymbol{P}})\, \varepsilon(\bar{\boldsymbol{P}})\right)\, \mathcal{V} \ .$$

Considering a subdomain $\mathcal{V}$ around a point $\boldsymbol{P}$ and approaching the limit as the volume of $\mathcal{V}$ approaches zero, $\bar{\boldsymbol{P}}$ coalesce into $\boldsymbol{P}$ and the **internal work by volume unit** is:

$$\mathcal{T}_{int} = \frac{d\mathcal{W}}{d\mathcal{V}} = \lim_{\mathcal{V}\to 0} \frac{\mathcal{W}}{\mathcal{V}} = Tr\, \left(\sigma\, \varepsilon\right) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \ .$$

The elementary internal work by unit volume provided by $\boldsymbol{\sigma}$ to increment the strains of a quantity $d\boldsymbol{\varepsilon}$ is:

$$d\mathcal{T}_{int} = \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} \ .$$

In a similar way, if $\overrightarrow{\boldsymbol{v}}$ denotes the velocity of the point $\boldsymbol{P}$, the elementary power of the surface force acting at a point $\boldsymbol{P}$ of the boundary, is:

$$d\mathcal{P}_s = \overrightarrow{\boldsymbol{dF}}_s \cdot \overrightarrow{\boldsymbol{v}} = \overrightarrow{\boldsymbol{t}} \cdot \overrightarrow{\boldsymbol{v}} \, d\mathcal{S} \ .$$

the elementary power of the volume force acting at a point $\boldsymbol{P}$ of $\mathcal{V}$ is:

$$d\mathcal{P}_v = \overrightarrow{\boldsymbol{dF}}_v \cdot \overrightarrow{\boldsymbol{v}} = \overrightarrow{\boldsymbol{f}} \cdot \overrightarrow{\boldsymbol{v}} \, d\mathcal{V} \ .$$

By arguments analogous to the ones used for the work, we conclude that the **internal power by volume unit** is:

$$\mathcal{P}_{int} = Tr \ (\sigma \, D) \ , \tag{8.5}$$

where we introduced the **strain velocity**:

$$D = \frac{1}{2} \left[\frac{\partial v}{\partial r} + \left(\frac{\partial v}{\partial r} \right)^T \right] = grad_s \, v \ . \tag{8.6}$$

8.3 Linear elasticity

8.3.1 Hooke's law

If the strains are small, many materials such as metals are elastic. We would like to generalize to bulky bodies Hooke's law 4.3 previously introduced for slender ones. Thus we claim that the elastic behaviour of the material is modeled by a linear map $\boldsymbol{E}$ mapping the 2-covariant strain tensor $\boldsymbol{\varepsilon}$ onto the 2-contravariant stress tensor $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = \boldsymbol{E}(\boldsymbol{\varepsilon}) \ . \tag{8.7}$$

Hence, there exists a 4-contravariant **elastic tensor $\boldsymbol{C}$** such that:

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} \ . \tag{8.8}$$

that reads with indicial notations:

$$\sigma^{ij} = C^{ijkl} \varepsilon_{kl} \ , \tag{8.9}$$

or alternatively :

$$\sigma = E(\varepsilon) \ , \tag{8.10}$$

where E is a linear map from the space $\mathbb{M}_{33}^{symm}$ of 3×3 matrices into itself. As the stress and strain tensors are both symmetric, the components of the elasticity tensor are subjected to the **minor symmetries**:

$$C^{ijkl} = C^{jikl} = C^{ijlk} \ .$$

Moreover, it is expected that the elastic behaviour is **reversible** in the following sense. For any loop $\mathcal{C}$ in the space of strain tensors:

$$\oint_{\mathcal{C}} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} = 0 \ , \tag{8.11}$$

where $\boldsymbol{\sigma}$ depends on $\boldsymbol{\varepsilon}$ through (8.9). In practice, this global condition is difficult to verify for every loop. In order to find an equivalent local condition, we consider a parallelogram of which the size approaches zero. Reasoning as in Subsection 3.3.2, we obtain:

$$\forall d\boldsymbol{\varepsilon}, \delta\boldsymbol{\varepsilon}, \qquad d\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} - \delta\boldsymbol{\sigma} : d\boldsymbol{\varepsilon} = 0 \ , \tag{8.12}$$

where $\boldsymbol{\sigma}$ is given by (8.9) and this condition is necessary and sufficient for (8.11). Using dummy indices, it reads:

$$d\sigma^{ij}\delta\varepsilon_{ij} - \delta\sigma^{kl}d\varepsilon_{kl} = (C^{ijkl} - C^{klij})d\varepsilon_{kl}\delta\varepsilon_{ij} = 0 \ .$$

Because the infinitesimal variations $d\boldsymbol{\varepsilon}$ and $\delta\boldsymbol{\varepsilon}$ are arbitrary, the components of the elasticity tensor are subjected to the **major symmetries**:

$$C^{ijkl} = C^{klij} \ .$$

Taking into account the minor and major symmetries, among the $3^4 = 81$ components C^{ijkl}, there are only $6\,(6+1)/2 = 21$ independant ones.

One of the interesting features of the reversible behaviour is the existence of a scalar **reversible energy potential** W generating the constitutive law, such that:

$$W(\boldsymbol{\varepsilon}) = W(\boldsymbol{\varepsilon}_0) + \int_{\boldsymbol{\varepsilon}_0}^{\boldsymbol{\varepsilon}} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} \ , \tag{8.13}$$

where $\boldsymbol{\varepsilon}_0$ is any reference strain and the integration path from $\boldsymbol{\varepsilon}_0$ to $\boldsymbol{\varepsilon}$ can be chosen arbitrarily. Indeed, $\mathcal{C}_1$ and $\mathcal{C}_2$ being to such paths, let us consider the loop $\mathcal{C}$ obtained by concatenation of $\mathcal{C}_1$ and $\mathcal{C}_2$. If the sense of $\mathcal{C}$ is the same as $\mathcal{C}_1$ and opposite to the one of $\mathcal{C}_2$, one has:

$$\oint_{\mathcal{C}} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} = \int_{\mathcal{C}_1} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} - \int_{\mathcal{C}_2} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} = 0 \ .$$

By differentiating (8.13), we recover the constitutive law:

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \ ,$$

which reads in indicial notation:

$$\sigma^{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \ .$$

Of course, the potential is definite at an arbitrary constant $W(\boldsymbol{\varepsilon}_0)$ which is note relevant for the constitutive law. Chosing $\boldsymbol{\varepsilon}_0 = \mathbf{0}$, $W(\mathbf{0}) = 0$ and the straight path from $\mathbf{0}$ to $\boldsymbol{\varepsilon}$, the integration is straightforward for Hooke's law (8.8) and gives:

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\,\boldsymbol{\varepsilon} : (\boldsymbol{C} : \boldsymbol{\varepsilon}) \ ,$$

or in indicial notations:

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\,C^{ijkl}\varepsilon_{ij}\varepsilon_{kl} \ .$$

8.3.2 Isotropic materials

It is worth to remark that the law (8.10) is the local representation in a given basis S of the intrinsic elasticity law (8.7). Latter on, we only consider the basis associated to reference coordinate systems in which the distances and other mechanical quantities are measured, the Galilean coordinate ones. Restricting us to orthonormal basis, we are working now with Euclidean tensors. Let us introduce now the class of materials with stronger symmetries:

Definition 8.2 *A material is* ***isotropic*** *if the behaviour of the material is the same in every Galilean coordinate system.*

In other words, we have in any orthonormal basis S':

$$\sigma' = E(\varepsilon') \ ,$$

As an orthonormal basis S is transformed into another one S' through an orthogonal transformation $R = S^{-1}S' \in \mathbb{O}(3)$, according to the transformation laws (8.4) and (7.5), the law is isotropic if:

$$\forall \varepsilon \in \mathbb{M}_{33}^{symm}, \qquad \forall R \in \mathbb{O}(3), \qquad R^T E(\varepsilon)\, R = E(R^T \varepsilon\, R) \ . \tag{8.14}$$

The general form of isotropic constitutive law of materials is given by **Rivlin-Ericksen representation theorem**:

Theorem 8.3 *The elasticity law is isotropic if and only the map E is of the form:*

$$E(\varepsilon) = \lambda_0(\iota(\varepsilon))\, 1_{\mathbb{R}^3} + \lambda_1(\iota(\varepsilon))\, \varepsilon + \lambda_2(\iota(\varepsilon))\, \varepsilon^2 \ . \tag{8.15}$$

Proof. The Demonstration is decomposed into three steps.

- **Step 1 :** *we prove that any matrix R which diagonalizes ε also diagonalizes* $E(\varepsilon)$. As the matrix ε is symmetric, it is diagonalizable and its eigenvalues ε_i are real numbers. Hence there exist an orthogonal transformation $R = (V_1, V_2, V_3)$ such that:
$$R^T \varepsilon\, R = diag(\varepsilon_1, \varepsilon_2, \varepsilon_2) \ . \tag{8.16}$$

Without loss of generality, we may assume that R is a rotation (otherwise, replace one of its column by its opposite). Besides, let us consider the **mirror symmetry** with respect to the plan Ox_2x_3:

$$M_1 = diag(-1,1,1) \ ,$$

which is clearly an orthogonal transformation. The relation (8.16) means that for $j = 1,2,3$ the column V_i of R is an eigenvector of ε corresponding to the eigenvalue ε_i. Hence we likewise have

$$(RM_1)^T \varepsilon \, (RM_1) = diag(\varepsilon_1, \varepsilon_2, \varepsilon_2) = R^T \varepsilon \, R \ .$$

since the effect of multiplying R by M_1 on the right is to replace its first column by its opposite. Owing to (8.14), we have:

$$M_1^T (R^T E(\varepsilon)\, R)\, M_1 = (RM_1)^T E(\varepsilon)\, (RM_1) = E((RM_1)^T \varepsilon \, (RM_1))$$
$$M_1^T (R^T E(\varepsilon)\, R)\, M_1 = E(R^T \varepsilon \, R) = R^T E(\varepsilon)\, R \ .$$

As a straighforward calculations shows, the effect to multiply the matrix $R^T E(\varepsilon)\, R$ by R^T on the left and R on the right is to cancel the elements of its first column (and first row) which are not on the diagonal. Next, we repeat this reasoning with the mirror symmetry M_2 with respect to the plan Ox_1x_3 that has the effect of cancelling the elements of the second column (and second row) of $R^T E(\varepsilon)\, R$ which are not on the diagonal. Then the matrix $R^T E(\varepsilon)\, R$ is diagonal. We just proved that any matrix R which diagonalizes ε also diagonalizes $E(\varepsilon)$.

- **Step 2 :** *we prove that the map E is necessarily of the form (8.15) where λ_i are real-valued functions.* Three cases must be distinguished. Assume first that the matrix ε has three distinct eigenvalues ε_i, with associated orthonomalized eigenvectors V_i. Then the two sets $\left\{1_{\mathbb{R}^3}, \varepsilon, \varepsilon^2\right\}$ and $\left\{V_1V_1^T, V_2V_2^T, V_3V_3^T\right\}$ span the same subspace of the linear space $\mathbb{M}_{33}^{symm}$. Indeed, we observe that:

$$1_{\mathbb{R}^3} = V_1V_1^T + V_2V_2^T + V_3V_3^T \ ,$$
$$\varepsilon = \varepsilon_1 V_1V_1^T + \varepsilon_2 V_2V_2^T + \varepsilon_3 V_3V_3^T \ ,$$
$$\varepsilon^2 = \varepsilon_1^2 V_1V_1^T + \varepsilon_2^2 V_2V_2^T + \varepsilon_3^2 V_3V_3^T \ ,$$

and that the van der Monde determinant

$$det \begin{pmatrix} 1 & 1 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ \varepsilon_1^2 & \varepsilon_2^2 & \varepsilon_3^2 \end{pmatrix} \ .$$

does not vanish, since the three eigenvalues are assumed to be distinct. Denoting by σ_i the eigenvalues of $\sigma = E(\varepsilon)$, the result of Step 1 shows that we can expand $E(\varepsilon)$ as:

$$E(\varepsilon) = \sigma_1 V_1V_1^T + \sigma_2 V_2V_2^T + \sigma_3 V_3V_3^T \ ,$$

and consequently also as:

$$E(\varepsilon) = \lambda_0(\varepsilon)\, 1_{\mathbb{R}^3} + \lambda_1(\varepsilon)\, \varepsilon + \lambda_2(\varepsilon)\, \varepsilon^2 \ , \tag{8.17}$$

where the components $\lambda_i(\varepsilon)$ are uniquely determined because the matrices $1_{\mathbb{R}^3}$, ε and ε^2 are linearly independents in this case.

Assume next the matrix ε has a double eigenvalue, let say $\varepsilon_2 = \varepsilon_3 \neq \varepsilon_1$. Then the two sets $\{1_{\mathbb{R}^3}, \varepsilon\}$ and $\left\{V_1V_1^T, V_2V_2^T + V_3V_3^T\right\}$ span the same subspace of the linear space $\mathbb{M}_{33}^{symm}$ since in this case:

$$1_{\mathbb{R}^3} = V_1V_1^T + (V_2V_2^T + V_3V_3^T), \qquad \varepsilon = \varepsilon_1 V_1V_1^T + \varepsilon_2(V_2V_2^T + V_3V_3^T) \ .$$

By a reasoning similar to the one of the first case, we prove that $E(\varepsilon)$ has also a double eigenvalue and we have:

$$E(\varepsilon) = \lambda_0(\varepsilon)\, 1_{\mathbb{R}^3} + \lambda_1(\varepsilon)\, \varepsilon \ . \tag{8.18}$$

Assume finally that the matrix ε has a triple eigenvalues. We likewize show that $E(\varepsilon)$ has also a triple eigenvalue and we have:

$$E(\varepsilon) = \lambda_0(\varepsilon)\, 1_{\mathbb{R}^3} \ . \tag{8.19}$$

it is worth noticing that (8.18) and (8.19) are particular case of (8.17) and we can use it further as general expression.

- **Step 3 :** *we observe that the λ_i are invariant functions of ε.* Indeed, combining the isotropy condition (8.14) and the expansion formula (8.17) gives:

$$\begin{aligned} E(R^T \varepsilon\, R) &= R^T \left[\lambda_0(R^T \varepsilon\, R)\, 1_{\mathbb{R}^3} + \lambda_1(R^T \varepsilon\, R)\, \varepsilon + \lambda_2(R^T \varepsilon\, R)\, \varepsilon^2\right] R \\ &= R^T E(\varepsilon)\, R = R^T \left[\lambda_0(\varepsilon)\, 1_{\mathbb{R}^3} + \lambda_1(\varepsilon)\, \varepsilon + \lambda_2(\varepsilon)\, \varepsilon^2\right] R \ . \end{aligned}$$

  Hence, $\lambda_i(R^T \varepsilon\, R) = \lambda_i(\varepsilon)$ because of the uniqueness of the expansion of $E(\varepsilon)$ in the spaces spanned by either sets $\left\{1_{\mathbb{R}^3}, \varepsilon, \varepsilon^2\right\}$, $\{1_{\mathbb{R}^3}, \varepsilon\}$ or $\{1_{\mathbb{R}^3}\}$, according to which case is considered. Then the invariance of the λ_i is satisfied if they depends on ε through the system $\iota(\varepsilon)$ of principal invariants. ■

We are in conditions to generalize Hooke's law 4.3 in the sense that stresses are proportional to corresponding deformations and the material is isotropic. According to the previous Theorem, the elastic law has the form (8.15). The last term containing the non linear factor ε^2, we cancel it by assuming $\lambda_2 = 0$. For the first term being linear, λ_0 must be linear then depending on ε through the only linear principal invariant $\iota_1(\varepsilon) = Tr(\varepsilon)$. Finally, the second term containing the factor ε, λ_1 must be constant. A straightforward calculation shows that the reversibility condition (8.12) is fulfilled. Hence, we must assume:

Law 8.4 *There exist two material constants* λ, μ *called* ***Lame's coefficients*** *such that the* ***isotropic linear elastic law*** *or* ***generalized Hooke's law*** *is reversible and given by:*

$$\sigma = \lambda \, Tr(\varepsilon) \, 1_{\mathbb{R}^3} + 2 \, \mu \, \varepsilon \; . \tag{8.20}$$

Of course, the relation (8.20) is only valid in Galilean coordinate systems. To find the free coordinate version of this law, let us remark that it reads in tensorial notations preserving the covariant and contravariant indices of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$:

$$\sigma^{ij} = \lambda \, (\delta^{kl} \varepsilon_{kl}) \, \delta^{ij} + 2 \, \mu \, \delta^{ik} \delta^{jl} \varepsilon_{kl} \; .$$

As the contravariant metric tensors $\breve{\boldsymbol{G}}^{-1}$ is represented by the identity matrix in Galilean coordinate systems, it is easy to guess the free coordinate form of Hooke's law using contracted products:

$$\boldsymbol{\sigma} = \lambda \, (\breve{\boldsymbol{G}}^{-1} : \boldsymbol{\varepsilon}) \, \breve{\boldsymbol{G}}^{-1} + 2 \, \mu \, \breve{\boldsymbol{G}}^{-1} \cdot \boldsymbol{\varepsilon} \cdot \breve{\boldsymbol{G}}^{-1} \; .$$

8.3.3 Elasticity problems

Chapter 9

Dynamics of 3D continua and elementary mechanics of fluids

9.1 Deformation and motion

Once again, we recover the extra dimension of time to work in the space-time with the convention 1.1 on indices.

Definition 9.1 *A body which is not rigid is called a* ***deformed body****. the material lengths and angles are not in general preserved by its motion. The motion of a deformed body is called a* ***deformation****.*

We model the deformations in two steps:

♢ Construct the group of coordinate changes wich preserve the uniform straight motions, the durations and the orientations of volumes (but not the distances, angles and volumes!). We denote it $\mathbb{GD}$ and we call it the **deformation group**.

♡ Determine the coordinate changes wich are admissible with $\mathbb{GD}$ in the sense that their Jacobian matrix belongs to $\mathbb{GD}$.

♢ In the first step, we wish to characterized the most simple deformations, those which are homogeneous, that is uniform on the body. It lies in the following theorem:

Theorem 9.2 *The coordinate changes for which are invariant:*

- *the uniform straight motions,*
- *the durations,*
- *the orientations of volumes,*

are regular affine maps of the following form:

$$dX = P\,dX' + C, \quad C = \begin{pmatrix} \tau \\ k \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ v & F \end{pmatrix}, \tag{9.1}$$

where $\tau \in \mathbb{R}$, $k \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ *and* F *is a matrix of* $\mathbb{GL}(3)$ *such that:*

$$\det(F) > 0 \ , \tag{9.2}$$

Their set is a group containing Galileo's one.

Proof. It is the same as Theorem 1.11 up to Formula (1.7). Taking into account that oriented volumes are transformed as:

$$V' = \det(F) V \ ,$$

their sign is preserved provided (9.2). The verification of the group structure of $\mathbb{GD}$ is straightforward. The Galilean transformations are particular deformations for which F is a rotation. ■

♡ Next, we wish to determine the coordinate change $X \mapsto X'$ such that:

$$\frac{\partial X}{\partial X'} = P = \begin{pmatrix} 1 & 0 \\ v & F \end{pmatrix} \in \mathbb{GD} \ , \tag{9.3}$$

This partial derivative system involves $(4 \times 4 = 16)$ equations for 4 unknowns (X^α). It is overdeterminated and has generally no solutions, excepted if the equations satisfy some compatibility conditions. We are faced to the same kind of problem than in Subsection 3.3.2 (excepted than the gravitation Γ is replaced by (9.3)). In order to find the compatibility conditions, we consider a parallelogram of which the size approaches zero. Reasoning likewise, we obtain:

$$\forall dX', \delta X', \qquad \delta P\, dX' - dP\, \delta X' = 0 \ , \tag{9.4}$$

where P is given by (9.3). In the following theorem, we introduce the column:

$$X' = \begin{pmatrix} t' \\ s' \end{pmatrix} \ . \tag{9.5}$$

Theorem 9.3 *Any coordinate change* $X \mapsto X'$ *of which the Jacobian matrix belongs to* $\mathbb{GD}$ *is compound of a smooth map:*

$$r = \varphi(t', s') \ , \tag{9.6}$$

and a clock change:

$$t = t' + \tau_0 \ . \tag{9.7}$$

Proof. Differentiating (9.3) provides:

$$\delta P\, dX' - dP\, \delta X' = \begin{pmatrix} 0 \\ \delta v\, dt' - dv\, \delta t' + \delta F\, ds' - dF\, \delta s' \end{pmatrix} .$$

The compatibility condition $\delta(dt') - d(\delta t') = 0$ is automatically fulfilled. Besides, denoting $d_{s'}F$ the infinesimal variation of F resulting from the variation ds', one gets:

$$\delta F\, ds' - dF\, \delta s' = \frac{\partial F}{\partial t'}(ds'\, \delta t' - \delta s'\, dt') + \delta_{s'} F\, ds' - d_{s'} F\, \delta s' \ .$$

Owing to (13.21) and after simplification, the second condition of compatibility reads:

$$\left(\frac{\partial v}{\partial s'} - \frac{\partial F}{\partial t'}\right)(\delta s'\, dt' - ds'\, \delta t') + (curl\, F^T)^T(\delta s' \times ds') = 0 \ .$$

The infinitesimal perturbations dX' and $\delta X'$ being arbitrary, this condition is satisfied if and only if:

$$\frac{\partial v}{\partial s'} = \frac{\partial F}{\partial t'}, \qquad curl\, F^T = 0 \ . \tag{9.8}$$

Under these conditions, the equation system (9.3) can be integrated, let:

$$\frac{\partial t}{\partial t'} = 1, \qquad \frac{\partial t}{\partial s'} = 0, \qquad \frac{\partial r}{\partial s'} = F, \qquad \frac{\partial r}{\partial t'} = v \ . \tag{9.9}$$

The integration of the two former equations leads to the clock change (9.7). The second equation is satisfied if and only if there exists an arbitrary column field $\varphi(t', s') \in \mathbb{R}^3$ such that:

$$F = \frac{\partial \varphi}{\partial s'} \ . \tag{9.10}$$

Introducing this last relation into the first condition (9.8), one gets:

$$\frac{\partial}{\partial s'}\left(v - \frac{\partial \varphi}{\partial t'}\right) = 0 \ .$$

There exists an arbitrary column $v_0(t') \in \mathbb{R}^3$ such that:

$$v = \frac{\partial \varphi}{\partial t'}(t', s') + v_0(t') \ .$$

Taking into account this last relation, (9.10) and the two latter equations of (9.60), we obtain after integration:

$$r = \varphi(t', s') + \varphi_0(t') \ ,$$

where φ_0 is a primitive of u_0. Without lost of generality, φ_0 may be absorbed in φ, leading to (9.6). ■

Of course, the rigid motions (5.3) are particular cases:

$$r = \varphi(t', s') = R(t')s' + r_0(t') \ .$$

For this reason, considering arbitrary deformations φ, s' will be called **Lagrangean** or **material coordinates**, by opposition to **Eulerian** or **spatial coordinates** r. People often choose φ such that:

$$s' = \varphi(0, s') \ ,$$

that allows to identify the Lagrangean coordinates with the initial position at $t = 0$ but this choice is not compulsory. Without lost of generality, we can forget in the sequel the change clock, putting $\tau_0 = 0$ and identifying t and t'.

On this ground, we whish to model the motion of 3D continua. Fixing s' in (9.6), we claim the map $t \mapsto \varphi(t, s')$ is the trajectory of the particle identified by s' and its velocity in the Eulerian representation is given by:

$$v = \frac{dr}{dt}\,|_{s'=C^{te}} = \frac{\partial \varphi}{\partial t} \ .$$

According to Theorem 9.2, the Jacobian matrix (9.3) is composed of a boost $u = v$ and a linear transformation F. Because of the implicit function theorem and taking into account (9.2) and (9.10), there exists a map $(t, r) \mapsto s' = \kappa(t, r)$ such that $r = \varphi(t, \kappa(t, r))$, at least locally. As s' is constant along the trajectory, $s' = \kappa(t, r)$ is an integral of the motion, thus in the Lagrangian representation:

$$v' = \frac{ds'}{dt} = \frac{\partial \kappa}{\partial t} + \frac{\partial \kappa}{\partial r}\,\frac{dr}{dt} = 0 \ , \tag{9.11}$$

or with simplified notations:

$$\frac{ds'}{dt} = \frac{\partial s'}{\partial t} + \frac{\partial s'}{\partial r}\, v = 0 \ ,$$

Introducing the 4-velocity vector represented by (1.12), it is worth to observe that it reads:

$$\frac{\partial s'}{\partial X}\, U = 0 \ . \tag{9.12}$$

More generally, we introduce:

Definition 9.4 *For any (scalar or vector) field* $(t, r) \mapsto q\,(t, r)$*, its* ***material derivative*** *or* ***total derivative*** *is:*

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \frac{\partial q}{\partial r}\, v = \frac{\partial q}{\partial X}\, U \ .$$

For that matter, it is interesting working directly in the space-time to model transport phenomena such as flows or heat transfer. For any scalar field q, the corresponding flux is represented by the 4-column $q\,U$. If it is free divergence:

$$div_X\,(q\,U) = \frac{\partial q}{\partial t} + div\,(q\,v) = 0\ , \tag{9.13}$$

q is conserved in the following sense. Integrating this relation on the volume $\mathcal{V}$ between the dates t_0 and t_1 and using Green formula (13.16) leads to:

$$\int_{\mathcal{V}} q\,d\mathcal{V}\ |_{t=t_1} = \int_{\mathcal{V}} q\,d\mathcal{V}\ |_{t=t_0} - \int_{t_0}^{t_1}\int_{\partial\mathcal{V}} q\,(v\cdot n)\,d\mathcal{V}\,dt\ .$$

Hence the total quantity of q on $\mathcal{V}$ at the final time t_1 is equal to the corresponding quantity at the initial time t_0 minus the quantity of q lost by transport at velocity v through the boundary of $\mathcal{V}$ between these two dates.

Definition 9.5 *The 3-column $q\,v$ measuring the rate of flow of a scalar q by unit area is called the* ***flux*** *of q. The corresponding space-time quantity is the 4-**flux** $q\,\vec{\boldsymbol{U}}$ of q. Condition (9.13) is the local expression of the* ***balance of*** *q.*

9.2 Flash-back: Galilean tensors

Reminding some remarks already done in Section 1.3.4, the set of Galilean transformations is a group (even if the word 'group' was not yet used). It is called **Galileo's group** and it is denoted $\mathbb{GAL}$ in the sequel. As subgroup of the affine group $\mathbb{A}ff(4)$, Galileo's group naturally acts onto the tensors by restriction their transformation laws. The $\mathbb{GAL}$-tensors are called **Galilean tensors**. In fact we already know such tensors, for instance the 4-**velocity** $\vec{\mathbf{U}}$ of which the components in Galilean coordinate systems are modified according to the transformation law (1.11) of a vector:

$$U = \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}\begin{pmatrix} 1 \\ v' \end{pmatrix}\ ,$$

which provides the velocity addition formula $v = u + R\,v'$. The 4-velocites are Galilean vectors. By the way, it is worth noting the following result:

Theorem 9.6 *Any non vanishing Galilean vector $\vec{\boldsymbol{V}}$ is the 4-flux of $q = V^0$ in any Galilean coordinate system.*

Proof. let $\vec{\boldsymbol{V}}$ be represented in a Galilean coordinate system X by the 4-column:

$$V = \begin{pmatrix} q \\ w \end{pmatrix}\ ,$$

where $q \in \mathbb{R}$ and $w \in \mathbb{R}^3$. According to (1.15), their transformation law (12.21) gives:

$$q' = q\ , \qquad w' = R^T(w - qu)\ . \tag{9.14}$$

Galilean transformations leave q invariant and there is no trouble to put q instead of q' in the sequel. By a method similar to the one of Section 3.1.1, we annihilate w' by choosing the Galilean boost $u = w/q$. Conversely, let us consider a Galilean coordinate system X' in which the vector $\vec{\boldsymbol{V}}$ has a reduced form:

$$V' = \begin{pmatrix} q \\ 0 \end{pmatrix} ,$$

In the spirit of the boost method initiated in Section 3.1.2, let X be another Galilean coordinate system obtained from X' through a Galilean boost v. Then, $V = q\,U$, that proves $\vec{\boldsymbol{V}}$ is the 4-flux of q. ■

If the 4-**flux** $q\,\vec{\mathbf{U}}$ is free covariant divergence:

$$\boldsymbol{Div}\,(q\,\vec{\boldsymbol{U}}) = 0 ,$$

Owing to (13.32) and (13.29), the reader can verify that the balance (9.13) of q is satisfied for a Galilean gravitation (3.38).

Let us pick up a Galilean coordinate system and let us consider a linear form represented by the key-row:

$$e^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} . \tag{9.15}$$

According to the transformation law (12.25), it is represented in any other Galilean coordinate system by the same key-row. We denote $\boldsymbol{e}^0$ this Galilean linear form and we call it the **time arrow**.

Likewise, let us consider a 2-contravariant linear tensor represented in a given coordinate system by the matrix:

$$\gamma = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathbb{R}^3} \end{pmatrix} . \tag{9.16}$$

According to the transformation law (13.3), it is represented in any other Galilean coordinate system by the same matrix and is denoted $\boldsymbol{\gamma}$. It is worth to remark that the contracted product of the two previous tensors, represented in a Galilean coordinate system by $e^0_\alpha \gamma^{\alpha\beta}$ vanishes:

$$\boldsymbol{e}^0 \cdot \boldsymbol{\gamma} = \boldsymbol{0} .$$

Also, let us consider an object $\boldsymbol{G}$ represented in any given Galilean coordinate system by the matrix:

$$G = \begin{pmatrix} -2\phi & A^T \\ A & 1_{\mathbb{R}^3} \end{pmatrix} , \tag{9.17}$$

where $\phi \in \mathbb{R}$ and $A \in \mathbb{R}^3$ are the potentials of the Galilean gravitation. According to their transformation law (6.16), $\boldsymbol{G}$ is a Galilean symmetric 2-covariant linear tensor because the transformation law (13.2) is satisfied.

Nevertheless, it is worth observing that $\boldsymbol{G}$ generally **is not a metric** –particularly in the simplest situation where there is no gravitation– because it is not necessarily nondegenerate.

The gravitation itself, although represented by Christoffel's symbols $\Gamma^{\alpha}_{\mu\beta}$, **is not a tensor** but is a covariant differential of which the transformation law is (13.27).

The study of the Dynamics of particles (Chapter 3) revealed an object called the linear 4-momentum, represented in a Galilean coordinate system by the column T gathering the components T^{α}. T being modified in a coordinate change according to (3.30), it is a vector. Now, have a look to the definition (3.37). We recognize the definition (13.26) of the covariant differential of a vector field. Hence $\boldsymbol{d}T$ is a nothing else as $\nabla_{dX} T$. It is not a simple change of notations unsofar as the covariant differential was introduced in Chapter 3 using heuristic arguments while the definition of Annex 13 is more rigorous and general. The advantage of the new viewpoint offered by the Tensor Analysis is to write the physical laws in an intrinsic form. For instance, the general equation of the motion provided by the law 3.11 is recast in a coordinate-free style as:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} \vec{\boldsymbol{T}} = \vec{\boldsymbol{H}} \ ,$$

where $\vec{\boldsymbol{T}}$ is the linear 4-momentum vector field and $\vec{\boldsymbol{H}}$ is the vector field representing the resultant of the other forces.

As the force torsor, the dynamical torsor $\boldsymbol{\mu}$ of a particle or a rigid body is a skew-symmetric 2-contravariant affine tensor, the transformation law (3.4) being nothing else (13.9). In the affine frame $(\boldsymbol{X}_0, (\vec{\boldsymbol{e}}_\alpha))$, it is decomposed as:

$$\boldsymbol{\mu} = T^{\alpha} \left(\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_\alpha - \vec{\boldsymbol{e}}_\alpha \otimes \boldsymbol{X}_0 \right) + J^{\alpha\beta}\, \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta \ .$$

When the transformation law is restricted to the Galilean transformations, it is a Galilean affine tensor. We call it Galilean torsor. Before going further, let us spend some time to calculate the covariant differential of a torsor considered as an affine tensor. Using the rule (13.42), one has:

$$\begin{aligned} \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{\mu} \ = \ & dT^{\alpha} \left(\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_\alpha - \vec{\boldsymbol{e}}_\alpha \otimes \boldsymbol{X}_0 \right) + T^{\alpha} \left(\boldsymbol{X}_0 \otimes \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \vec{\boldsymbol{e}}_\alpha - \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \vec{\boldsymbol{e}}_\alpha \otimes \boldsymbol{X}_0 \right) \\ & + T^{\alpha} \left(\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_\alpha - \vec{\boldsymbol{e}}_\alpha \otimes \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{X}_0 \right) + dJ^{\alpha\beta}\, \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta \\ & + J^{\alpha\beta} \left(\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta + \vec{\boldsymbol{e}}_\alpha \otimes \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \vec{\boldsymbol{e}}_\beta \right) \ . \end{aligned} \tag{9.18}$$

Taking into account the infinitesimal motion of the basis vectors (13.28) and of the origin (13.39), it holds:

$$\begin{aligned} \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{\mu} \ = \ & dT^{\alpha} \left(\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_\alpha - \vec{\boldsymbol{e}}_\alpha \otimes \boldsymbol{X}_0 \right) + \Gamma^{\rho}_{\alpha} T^{\alpha} \left(\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_\rho - \vec{\boldsymbol{e}}_\rho \otimes \boldsymbol{X}_0 \right) \\ & + \Gamma^{\beta}_{A} T^{\alpha} \left(\vec{\boldsymbol{e}}_\beta \otimes \vec{\boldsymbol{e}}_\alpha - \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta \right) + dJ^{\alpha\beta}\, \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta \\ & + \Gamma^{\rho}_{\alpha} J^{\alpha\beta}\, \vec{\boldsymbol{e}}_\rho \otimes \vec{\boldsymbol{e}}_\beta + \Gamma^{\rho}_{\beta} J^{\alpha\beta}\, \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\rho \ , \end{aligned} \tag{9.19}$$

and, by renaming the dummy indices, we obtain:

$$\tilde{\nabla}_{\overrightarrow{dX}}\,\boldsymbol{\mu} = \tilde{\nabla}_{dX}T^{\beta}\,(\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_{\beta} - \vec{\boldsymbol{e}}_{\beta} \otimes \boldsymbol{X}_0) + \tilde{\nabla}_{dX}J^{\alpha\beta}\,\vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\beta}\ , \tag{9.20}$$

with:

$$\tilde{\nabla}_{dX}T^{\beta} = dT^{\beta} + \Gamma^{\beta}_{\rho}T^{\rho}\ , \tag{9.21}$$

$$\tilde{\nabla}_{dX}J^{\alpha\beta} = dJ^{\alpha\beta} + \Gamma^{\alpha}_{\rho}J^{\rho\beta} + \Gamma^{\beta}_{\rho}J^{\alpha\rho} + \Gamma^{\alpha}_{A}T^{\beta} - T^{\alpha}\Gamma^{\beta}_{A}\ . \tag{9.22}$$

In matrix form, the covariant differential of a torsor field is given by:

$$\tilde{\nabla}_{dX}T = dT + \Gamma(dX)\,T, \qquad ,$$

$$\tilde{\nabla}_{dX}J = dJ + \Gamma(dX)\,J + J\,(\Gamma(dX))^{T} + \Gamma_{A}(dX)\,T^{T} - T\,(\Gamma_{A}(dX))^{T}\ ,$$

We recover in a more rigorous framework the covariant differential (5.42) of the dynamical torsor of a rigid body introduced in an heuristic way. Introducing the resultant torsor of the other forces (i.e. different from the gravitation):

$$\boldsymbol{\mu}^{\star} = H^{\alpha}\,(\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_{\alpha} - \vec{\boldsymbol{e}}_{\alpha} \otimes \boldsymbol{X}_0) + G^{\alpha\beta}\,\vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\beta}\ ,$$

the generalized equation of rigid body motion given by the law 5.11 can be recast in a coordinate-free style as:

$$\tilde{\nabla}_{\overrightarrow{dX}}\,\boldsymbol{\mu} = \boldsymbol{\mu}^{\star}\ .$$

In this regard, it is worth to recall that, according to Subsection 5.3.2, the map $\boldsymbol{A}$ of Subsection 13.5.3 was choosen as the identity. Thus putting $A = 1_{\mathbb{R}^4}$ in (13.38), we recover (5.54) in the form:

$$\Gamma_{A}(dX) = dX - \nabla_{dX}C\ .$$

Coming back to tensor notations, (13.41) reads:

$$\Gamma^{\alpha}_{A\beta} = \delta^{\alpha}_{\beta} - \nabla_{\beta}C^{\alpha}\ . \tag{9.23}$$

9.3 Dynamical torsor of a 3D continuum

We would like to define the dynamical torsor of a continuum. Imitating the model of the Statics developped in the previous Chapter, we extend the notion of torsor in a space-time framework under the form of a vector valued torsor, according to the Definition 7.5. In an affine frame $(\boldsymbol{X}_0, (\vec{\boldsymbol{e}}_{\alpha}))$, it is decomposed as:

$$\boldsymbol{\mu} = \mu^{\gamma}\vec{\boldsymbol{e}}_{\gamma}, \qquad \mu^{\gamma} = T^{\beta\gamma}\,(\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_{\beta} - \vec{\boldsymbol{e}}_{\beta} \otimes \boldsymbol{X}_0) + J^{\alpha\beta\gamma}\,\vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\beta}\ .$$

Let $(\boldsymbol{X}'_0, (\vec{\boldsymbol{e}}'_{\alpha}))$ be a new affine frame obtained from the old one through an affine transformation $a = (C, P)$. Hence, the transformation law of the torsor is:

$$T'^{\rho\sigma} = (P^{-1})^{\rho}_{\beta}(P^{-1})^{\sigma}_{\gamma}T^{\beta\gamma}\ , \tag{9.24}$$

$$J'^{\rho\sigma\tau} = \left[(P^{-1})^\rho_\alpha (P^{-1})^\sigma_\beta J^{\alpha\beta\gamma} + C'^\rho \left\{ (P^{-1})^\sigma_\beta T^{\beta\gamma} \right\} - \left\{ (P^{-1})^\rho_\beta T^{\beta\gamma} \right\} C'^\sigma \right] (P^{-1})^\tau_\gamma \ . \tag{9.25}$$

with: $C' = -P^{-1} C$.

Next, let us calculate the covariant differential of the continuum:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{\mu} = \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} (\mu^\gamma \vec{\boldsymbol{e}}_\gamma) = (\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \mu^\gamma) \vec{\boldsymbol{e}}_\gamma + \mu^\gamma (\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \vec{\boldsymbol{e}}_\gamma) \ .$$

Taking into account (13.28), we have:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{\mu} = (\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \mu^\gamma + \Gamma^\gamma_\rho \mu^\rho) \vec{\boldsymbol{e}}_\gamma \ .$$

Calculating the first term of the right hand member is similar to the one (9.20) of the scalar valued dynamical torsor (add indices γ in (9.20), (9.21) and (9.22)). Finally we obtain:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{\mu} = \left[\tilde{\nabla}_{dX} T^{\beta\gamma} (\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_\beta - \vec{\boldsymbol{e}}_\beta \otimes \boldsymbol{X}_0) + \tilde{\nabla}_{dX} J^{\alpha\beta\gamma} \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta \right] \vec{\boldsymbol{e}}_\gamma \ ,$$

with:

$$\tilde{\nabla}_{dX} T^{\beta\gamma} = dT^{\beta\gamma} + \Gamma^\beta_\rho T^{\rho\gamma} + \Gamma^\gamma_\rho T^{\beta\rho} \ ,$$

$$\tilde{\nabla}_{dX} J^{\alpha\beta\gamma} = dJ^{\alpha\beta\gamma} + \Gamma^\alpha_\rho J^{\rho\beta\gamma} + \Gamma^\beta_\rho J^{\alpha\rho\gamma} + \Gamma^\gamma_\rho J^{\alpha\beta\rho} + \Gamma^\alpha_A T^{\beta\gamma} - T^{\alpha\gamma} \Gamma^\beta_A \ .$$

It is worth to recall that coefficients Γ^α_ρ represent the gravitation, as discussed at length in Chapter 3 devoted to the Dynamics. Hence, there exists a field $\tilde{\nabla} \boldsymbol{\mu}$ of 1-covariant and 3-contravariant affine tensors such that:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{\mu} = (\tilde{\nabla} \boldsymbol{\mu}) \cdot \overrightarrow{d\boldsymbol{X}} \ .$$

Using Christoffel's symbols (13.30) and additional symbols (13.40), one has:

$$\tilde{\nabla} \boldsymbol{\mu} = \left[\tilde{\nabla}_\sigma T^{\beta\gamma} (\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_\beta - \vec{\boldsymbol{e}}_\beta \otimes \boldsymbol{X}_0) + \tilde{\nabla}_\sigma J^{\alpha\beta\gamma} \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta \right] \vec{\boldsymbol{e}}_\gamma \otimes \boldsymbol{e}^\sigma \ ,$$

with:

$$\tilde{\nabla}_\sigma T^{\beta\gamma} = \frac{\partial T^{\beta\gamma}}{\partial X^\sigma} + \Gamma^\beta_{\sigma\rho} T^{\rho\gamma} + \Gamma^\gamma_{\sigma\rho} T^{\beta\rho} \ ,$$

$$\tilde{\nabla}_\sigma J^{\alpha\beta\gamma} = \frac{\partial J^{\alpha\beta\gamma}}{\partial X^\sigma} + \Gamma^\alpha_{\sigma\rho} J^{\rho\beta\gamma} + \Gamma^\beta_{\sigma\rho} J^{\alpha\rho\gamma} + \Gamma^\gamma_{\sigma\rho} J^{\alpha\beta\rho} + \Gamma^\alpha_{A\sigma} T^{\beta\gamma} - T^{\alpha\gamma} \Gamma^\beta_{A\sigma} \ .$$

By contraction, we define the **covariant divergence** of the dynamical tensor of the continuum:

$$\tilde{\boldsymbol{Div}}\, \boldsymbol{\mu} = \tilde{\nabla}_\gamma T^{\beta\gamma} (\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_\beta - \vec{\boldsymbol{e}}_\beta \otimes \boldsymbol{X}_0) + \tilde{\nabla}_\gamma J^{\alpha\beta\gamma} \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta \ . \tag{9.26}$$

with:

$$\tilde{\nabla}_\gamma T^{\beta\gamma} = \frac{\partial T^{\beta\gamma}}{\partial X^\gamma} + \Gamma^\beta_{\gamma\rho} T^{\rho\gamma} + \Gamma^\gamma_{\gamma\rho} T^{\beta\rho} \ , \tag{9.27}$$

$$\tilde{\nabla}_\gamma J^{\alpha\beta\gamma} = \frac{\partial J^{\alpha\beta\gamma}}{\partial X^\gamma} + \Gamma^\alpha_{\gamma\rho} J^{\rho\beta\gamma} + \Gamma^\beta_{\gamma\rho} J^{\alpha\rho\gamma} + \Gamma^\gamma_{\gamma\rho} J^{\alpha\beta\rho} + \Gamma^\alpha_{A\gamma} T^{\beta\gamma} - T^{\alpha\gamma} \Gamma^\beta_{A\gamma} \ . \tag{9.28}$$

At this stage, have a break to look back to the Static of 3D continua (just a matter to erase the time dimension). In the absence of gravitation, Christoffel's symbols vanish and we recover the covariant affine divergence of the torsor field given by (7.18) and (7.19).

We are now able to generalize the local equation (7.17) to the Dynamics of 3D continua. Although there is *a priori* a degree of arbitrariness in this choice, we adopt the biais to consider a **generalized Cauchy medium**, claiming that:

- There exists a field of vectors $\boldsymbol{X} \mapsto \vec{\boldsymbol{H}}(\boldsymbol{X})$ representing the resultant of other forces (i.e. different from the gravitation) and its torsor is:

$$\boldsymbol{\mu}_{\vec{\boldsymbol{H}}} = \boldsymbol{X} \otimes \vec{\boldsymbol{H}}(\boldsymbol{X}) - \vec{\boldsymbol{H}}(\boldsymbol{X}) \otimes \boldsymbol{X} \ ,$$

$$\boldsymbol{\mu}_{\vec{\boldsymbol{H}}} = H^{\beta} \left(\boldsymbol{X}_0 \otimes \vec{\boldsymbol{e}}_{\beta} - \vec{\boldsymbol{e}}_{\beta} \otimes \boldsymbol{X}_0\right) + G^{\alpha\beta} \, \vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\beta} \ , \tag{9.29}$$

  with:

$$G^{\alpha\beta} = X^{\alpha} H^{\beta} - H^{\alpha} X^{\beta} \ , \tag{9.30}$$

- There exists a field of linear 2-contravariant tensors $\boldsymbol{X} \mapsto \boldsymbol{T}(\boldsymbol{X})$. The corresponding vector valued torsor $\boldsymbol{\mu}_{\boldsymbol{T}}$ is such that for any covector $\boldsymbol{N}$:

$$\boldsymbol{N}(\boldsymbol{\mu}_{\boldsymbol{T}}) = \boldsymbol{X} \otimes (\boldsymbol{T} \cdot \boldsymbol{N}) - (\boldsymbol{T} \cdot \boldsymbol{N}) \otimes \boldsymbol{X} \ . \tag{9.31}$$

- The motion of the 3D continuum is governed by the free-coordinate equation:

$$\boxed{\tilde{\boldsymbol{Div}} \, \boldsymbol{\mu}_{\boldsymbol{T}} + \boldsymbol{\mu}_{\vec{\boldsymbol{H}}} = \boldsymbol{0}} \tag{9.32}$$

These assumptions are justified insofar as the predictions agree with a very wide spectrum of experimental observations. We start with a straightforward result:

Theorem 9.7 *A torsor satisfies (9.31) if and only if its components $J^{\alpha\beta\gamma}$ in the affine frame $(\boldsymbol{X}, (\vec{\boldsymbol{e}}_{\alpha}))$ vanish.*

Proof. Using the transformation laws (9.24) and (9.25), we argue by analogy with the proof of Theorem 7.6. ■

Next, we consider the simple case where:

- we put $C^{\alpha} = 0$ in (9.23), that amounts to restrict ourself to proper coordinate systems (in the sense of Subsection 3.1.2)) where the elementary volum is at rest,

- we are working in the affine frame $(\boldsymbol{X}, (\vec{\boldsymbol{e}}_{\alpha}))$, then $J^{\alpha\beta\gamma}$ components of $\boldsymbol{\mu}_{\boldsymbol{T}}$ and $G^{\alpha\beta}$ components of $\boldsymbol{\mu}_{\vec{\boldsymbol{H}}}$ vanish!

Taking into account (9.28), the law (9.32) leads to:

$$\boxed{T^{\beta\alpha} - T^{\alpha\beta} = 0 \ .} \tag{9.33}$$

We leave it to the reader to verify that it is true in every affine frame, owing to the transformation law (9.24).

9.4 The stress-mass tensor

9.4.1 Transformation law and invariants

Decomposing (9.31) in an affine frame, we verify that:

$$J^{\alpha\beta\gamma} = X^\alpha T^{\beta\gamma} - T^{\alpha\gamma} X^\beta \ .$$

Hence the structure of the dynamical torsor of the continuum is fixed by the one of the $T^{\alpha\gamma}$ components of the 2-contravariant linear tensor $\boldsymbol{T}$. For sake of easiness, they are gathered into a symmetric matrix T. Their transformation law (9.24) is given in matrix form by (13.3):

$$T' = P^{-1} T \, P^{-T} \ . \tag{9.34}$$

Taking into account the structure of the space-time, T is decomposed by blocks:

$$T = \begin{pmatrix} \rho & p^T \\ p & -\sigma_\star \end{pmatrix} \ ,$$

where ρ is a scalar, $p \in \mathbb{R}^3$ and $\sigma_\star \in \mathbb{M}_{33}^{symm}$. From now on, we consider $\boldsymbol{T}$ as a Galilean tensor by restriction of its transformation law to Galilean transformations. Owing to their decomposition by block (1.15), the transformation law (9.34) itemizes in:

$$\boxed{\rho' = \rho \ ,} \tag{9.35}$$

$$\boxed{p' = R^T (p - \rho\, u) \ ,} \tag{9.36}$$

$$\boxed{\sigma'_\star = R^T \left(\sigma_\star + u\, p^T + p\, u^T - \rho\, u\, u^T \right) R \ .} \tag{9.37}$$

It is worth to observe that Galilean transformations preserve the component ρ and there is no trouble to put ρ instead of ρ' in the sequel. To find the other invariants of $\boldsymbol{T}$, we follow the method applied in Section 3.1.1 to the dynamical torsor of a particle. We discuss only the case that the invariant component ρ does not vanish. Starting in any Galilean coordinate system X, we choose the Galilean boost:

$$u = \frac{p}{\rho} \ ,$$

which annihilates p' and reduces (9.37) to:

$$\sigma'_\star = R^T \left(\sigma_\star + \frac{1}{\rho}\, p\, p^T \right) R \ . \tag{9.38}$$

This suggests to cast a glance to the matrix:

$$\sigma = \sigma_\star + \frac{1}{\rho}\, p\, p^T \ . \tag{9.39}$$

Taking into account (9.35), (9.36) and (9.37), we obtain its transformation law:

$$\sigma' = R^T \sigma\, R \ . \tag{9.40}$$

As the matrix σ is symmetric, it is diagonalizable and its eigenvalues $\sigma_1, \sigma_2, \sigma_3$ are real numbers and the missing invariants of $\boldsymbol{T}$.

9.4.2 Boost method

Conversely, let us consider a Galilean coordinate system in which the tensor field $\boldsymbol{T}$ at a given point of coordinates X' has a **reduced form**:

$$T' = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -\sigma_1 & 0 & 0 \\ 0 & 0 & -\sigma_2 & 0 \\ 0 & 0 & 0 & -\sigma_3 \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & -\sigma' \end{pmatrix} \ ,$$

where the diagonal matrix σ' containing the σ_i is the matrix (9.39) in the coordinates X'. In the spirit of the boost method initiated in Subsection 3.1.2, we claim now the elementary volume around the point r' is at rest at time t'. Let X be another Galilean coordinate system obtained from X' through a Galilean boost v combined with a rotation R. Applying the inverse transformation law of (9.34):

$$T = P\, T'\, P^T \ , \tag{9.41}$$

we obtain:

$$p = \rho\ v, \qquad \sigma_\star = R\, \sigma'\, R^T - \rho\, v\, v^T \ .$$

The boost method turns out the physical meaning of the components:

- by analogy with Subsection 3.1.2, the invariant quantity ρ is interpreted as the mass per unit volume or **density**.

- The quantity p, product of the density and the velocity, is the **linear momentum** (per unit volume).
- Eliminating the velocity v between the two previous relations and taking into account (9.40), we recover (9.39). Besides, we recognize in (9.40) the transformation law (7.5) of Euclidean stress tensors. Thus σ can be identified to the **statical stresses**, while:

$$\sigma_\star = \sigma - \rho\, v\, v^T \ ,$$

are the **dynamical stresses**.

Definition 9.8 *The* ***stress-mass tensor*** $\boldsymbol{T}$ *is structured into three components:*

- *the* ***density*** ρ,
- *the* ***linear momentum*** p,
- *the* ***dynamical stresses*** $\sigma_\star$.

The invariants of the stress-mass tensor are:

- *the* ***density*** ρ,
- *the* ***principal stresses*** $\sigma_1, \sigma_2, \sigma_3$.

In matrix form, the stress-mass tensor reads:

$$\boxed{T = \begin{pmatrix} \rho & p^T \\ p & -\sigma_\star \end{pmatrix} = \begin{pmatrix} \rho & \rho\, v^T \\ \rho\, v & \rho\, v\, v^T - \sigma \end{pmatrix} \ .} \tag{9.42}$$

Before going further, let us stop to have adapt the boost method according to the motion of the particles modelized in Section 9.1. Let us consider the coordinate system X' associated to the Lagrangian representation, given by (9.5). It is not generally Galilean. Because of (9.11), the stress-mass tensor reads in the Lagrangian representation :

$$T' = \begin{pmatrix} \rho' & 0 \\ 0 & -\sigma' \end{pmatrix} \ , \tag{9.43}$$

but the matrix σ' of **material stresses** is not in general diagonal. Applying the inverse transformation law (9.34) with the boost $u = v$, we obtain:

$$\rho = \rho', \qquad p = \rho\ v, \qquad \sigma_\star = \sigma - \rho\, v\, v^T \ , \tag{9.44}$$

where **spatial stresses** σ in the Eulerian representation are related to the material stresses according to:

$$\sigma = F\, \sigma'\, F^T \ . \tag{9.45}$$

9.5 Euler's equations of motion

We claimed that the motion of 3D continua are (9.32) which, taking into account (9.26) are structured into two groups:

$$\tilde{\nabla}_\gamma T^{\beta\gamma} + H^\beta = 0, \qquad \tilde{\nabla}_\gamma J^{\alpha\beta\gamma} + G^{\alpha\beta} = 0 \ . \tag{9.46}$$

The second group led to the symmetry conditions (9.33) of the stress-mass tensor. It remais to examine the consequences of the first group, owing to the structure (9.42) of the stress-mass tensor in Galilean coordinate systems which reads in tensor notations:

$$T^{00} = \rho, \qquad T^{0j} = T^{j0} = \rho\, v^j, \qquad T^{ij} = \sigma^{ij} - \rho\, v^i v^j \ .$$

Excluding thrusts, we assume that the other forces (different from the gravitation) are modeled by 4-column H of the form (3.75):

$$H = \begin{pmatrix} 0 \\ -f \end{pmatrix} \ ,$$

where f is the volume force introduced by Definition (7.2). In indicial notations, we have:

$$H^0 = 0, \qquad H^j = -f^j \ .$$

Besides, the gravitation being represented in a Galilean coordinate system by (3.38), the non vanishing Christoffel's symbols are:

$$\Gamma^j_{00} = -g^j, \qquad \Gamma^j_{0k} = \Gamma^j_{k0} = \Omega^j_k \ , \tag{9.47}$$

where Ω^j_k is a simplified notation for the elements of the skew-symmetric matrix $j(\Omega)$ defined by (12.8). Taking into account (9.27), the first group of (9.46) reads:

$$\tilde{\nabla}_\gamma T^{\beta\gamma} + H^\beta = \frac{\partial T^{\beta\gamma}}{\partial X^\gamma} + \Gamma^\beta_{\gamma\rho} T^{\rho\gamma} + \Gamma^\gamma_{\gamma\rho} T^{\beta\rho} + H^\beta = 0 \ . \tag{9.48}$$

By putting $\beta = 0$ which corresponds the time coordinate, we obtain:

$$\tilde{\nabla}_\gamma T^{0\gamma} + H^0 = \frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0j}}{\partial r^j} = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r^j}(\rho\, v^j) = 0 \ . \tag{9.49}$$

For the spatial coordinates, we put $\beta = i$ that gives, taking into account the vanishing terms:

$$\tilde{\nabla}_\gamma T^{i\gamma} + H^i = \frac{\partial T^{i0}}{\partial t} + \frac{\partial T^{ij}}{\partial r^j} + \Gamma^i_{00} T^{00} + \Gamma^i_{0k} T^{k0} + \Gamma^i_{k0} T^{0k} - f^i = 0 \ .$$

$$\tilde{\nabla}_\gamma T^{i\gamma} + H^i = \frac{\partial}{\partial t}(\rho\, v^i) + \frac{\partial}{\partial r^j}(\rho\, v^i v^j - \sigma^{ij}) - \rho\, g^i + 2\rho\, \Omega^i_k\, v^k - f^i = 0 \ ,$$

or, after differentiation and rearrangement:

$$\rho\,\frac{\partial v^i}{\partial t} + \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r^j}(\rho\, v^j)\right]\, v^i + \rho\, v^j\,\frac{\partial v^i}{\partial r^j} = \frac{\partial \sigma^{ij}}{\partial r^j} + f^i + \rho\,(g^i - 2\Omega^i_k\, v^k)\ .$$

Taking into account (9.51) leads to:

$$\rho\,\frac{\partial v^i}{\partial t} + \rho\, v^j\,\frac{\partial v^i}{\partial r^j} = \frac{\partial \sigma^{ij}}{\partial r^j} + f^i + \rho\,(g^i - 2\Omega^i_k\, v^k)\ . \tag{9.50}$$

In short, the first group of (9.46) is recast as (9.49) and (9.50) that can be formulated in matrix notation as follows.

Law 9.9 *The motion of a 3D continuum obeys **Euler's equations**:*

♢ ***balance of mass***

$$\boxed{\frac{\partial \rho}{\partial t} + div\,(\rho\, v) = 0} \tag{9.51}$$

♡ ***balance of linear momentum***

$$\boxed{\rho\,\left[\frac{\partial v}{\partial t} + \frac{\partial v}{\partial r}\, v\right] = (div\ \sigma)^T + f + \rho(g - 2\ \Omega \times v)} \tag{9.52}$$

That requires some comments:

- This formulation is consistent with Galileo's principle of relativity 1.13 in the sense that the form of these equations is the same in all the Galilean coordinate systems. It is ensured particularly thanks to the last term in (9.52).

♢ Comparing (9.35), (9.36) and (9.14) shows that:

$$N = \begin{pmatrix} \rho \\ p \end{pmatrix}\ ,$$

represents a Galilean vector $\vec{\boldsymbol{N}}$ and, according to Theorem 9.6, it is the 4-flux of mass density ρ:

$$\vec{\boldsymbol{N}} = \rho\,\vec{\boldsymbol{U}}\ .$$

Equation (9.51) means that this field is divergence free:

$$\boldsymbol{Div}\,\vec{\boldsymbol{N}} = 0\ ,$$

Taking into account Definition 9.4 of the material derivative and (12.38), it is worth to observe also that it can read:

$$\frac{\partial \rho}{\partial t} + div\ (\rho\ v) = \frac{d\rho}{dt} + \rho\ div\ v = 0\ . \tag{9.53}$$

We wish to show it traduces the balance of mass. Let ρ_0 be the value of the density at the date $t = 0$ of the particle identified by the Lagrangian coordinates s'. Then, ρ_0 is a function of s'. As ρ is mass by unit volume, the actual value at the date t is

$$\rho = \frac{\rho_0}{det\ F} = \rho_0\ det\ \left(\frac{\partial s'}{\partial r}\right) \tag{9.54}$$

Next, we need the following proposition:

Lemma 9.10 *For every motion, one has:*

$$\frac{d}{dt}\ \left(\frac{\partial s'}{\partial r}\right) + \frac{\partial s'}{\partial r}\ \frac{\partial v}{\partial r} = 0\ . \tag{9.55}$$

Proof. Let δX be any uniform vector field on $\mathcal{M}$ and U be the 4-velocity. Differentiating (9.12) gives

$$\frac{\partial}{\partial X}\left(\frac{\partial s'}{\partial X}\ U\right)\ \delta X = \delta\ \left(\frac{\partial s'}{\partial X}\ U\right) = \delta\ \left(\frac{\partial s'}{\partial X}\right)\ U + \frac{\partial s'}{\partial X}\ \delta U = 0$$

$$= \frac{\partial}{\partial X}\ (\delta s')\, U + \frac{\partial s'}{\partial X}\ \frac{\partial U}{\partial X}\ \delta X = \frac{\partial}{\partial X}\ \left(\frac{\partial s'}{\partial X}\ \delta X\right)\ U + \frac{\partial s'}{\partial X}\ \frac{\partial U}{\partial X}\ \delta X = 0$$

Consequently

$$\frac{d}{dt}\ \left(\frac{\partial s'}{\partial X}\ \delta X\right) = \frac{\partial}{\partial X}\ \left(\frac{\partial s'}{\partial X}\ \delta X\right)\ U = -\frac{\partial s'}{\partial X}\ \frac{\partial U}{\partial X}\ \delta X$$

As δX is uniform, one has

$$\frac{d}{dt}\ \left(\frac{\partial s'}{\partial X}\right)\ \delta X = -\frac{\partial s'}{\partial X}\ \frac{\partial U}{\partial X}\ \delta X$$

As δX is arbitrary, it holds

$$\frac{d}{dt}\ \left(\frac{\partial s'}{\partial X}\right) = -\frac{\partial s'}{\partial X}\ \frac{\partial U}{\partial X}$$

or, in details

$$\frac{d}{dt}\ \left(\ \frac{\partial s'}{\partial t}\quad \frac{\partial s'}{\partial r}\ \right) = -\left(\ \frac{\partial s'}{\partial t}\quad \frac{\partial s'}{\partial r}\ \right)\begin{pmatrix} 0 & 0 \\ & \\ \dfrac{\partial\ v}{\partial t} & \dfrac{\partial\ v}{\partial r}\end{pmatrix}$$

that proves (9.55). ■

Then, we prove the following proposition:

Theorem 9.11 *If ρ_0 is a smooth function of s', then the density of mass*

$$\rho = \rho_0 \ (s') \ det \ \left(\frac{\partial s'}{\partial r} \right) \ , \tag{9.56}$$

verifies the balance of mass (9.51).

Proof. Differentiating (9.56), one has

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial s'} \ \frac{ds'}{dt} + Tr \ \left(\frac{\partial \rho}{\partial \left(\frac{\partial s'}{\partial r} \right)} \ \frac{d}{dt} \ \left(\frac{\partial s'}{\partial r} \right) \right)$$

or, owing to (9.11) and (13.24):

$$\frac{d\rho}{dt} = \rho \ Tr \ \left(\frac{\partial r}{\partial s'} \ \frac{d}{dt} \ \left(\frac{\partial s'}{\partial r} \right) \right) \tag{9.57}$$

Then, introducing (9.55) into (9.57), it holds

$$\frac{d\rho}{dt} = -\rho \ Tr \ \ (grad \ v \ ) = -\rho \ div \ v$$

that proves the balance of mass (9.51). ■

♡ Applying Definition 9.4 of the material derivative, Equation (9.52) is reduced to:

$$\boxed{\rho \ \frac{dv}{dt} = (div \ \sigma)^T + f + \rho(g - 2 \ \Omega \times v) \ .} \tag{9.58}$$

It clearly appears as representing the balance of linear momentum for a unit volume because the product of the density of mass by the acceleration dv/dt is equal to the sum of the gravitational forces, the other external forces f and the internal forces given by the divergence of the static stresses.

Another relevant equation is the balance of energy, a simple consequence of the one of the linear momentum. By taking the scalar product of both members of (9.58) by v, we have:

$$\rho \, \frac{dv}{dt} \cdot v = (div \ \sigma) \ v + (f + \rho g) \cdot v \ ,$$

the scalar triple product in the last term disapearing because v occurs twice in it. Owing to (13.12) and the symmetry of σ, we obtain the **balance of energy**:

$$\boxed{\rho \, \frac{d}{dt} \left(\frac{1}{2} \ \| \ v \ \|^2 \right) = div \, (\sigma v) - Tr \, (\sigma D) + (f + \rho g) \cdot v \ ,} \tag{9.59}$$

where occurs the strain velocity (8.6). This equation teach us that the time rate of kinetic energy is balanced by the divergence of the **stress transport** σv, the opposite of the internal power by volume unit, the power of the volume forces and of the gravity.

9.6 Constitutive laws in Dynamics

A **constitutive law** is a relation describing the mechanical behaviour of a material (solid or fluid). We already know the most simple of them, Hooke's law, but there is a very large spectrum of phenomenological behaviours that cannot be deduced from simple axioms of the statics or dynamics of continua. Their accurate description requires additional information resulting from experimental testing. Nevertheless, it is possible to determine general conditions satisfied by them in order to be consistent with Galileo's principle of relativity.

If the strains are small, we claimed that for elastic bodies in Statics, the stress tensor representing the internal forces is proportional to the strain tensor associated to the motion of the continuum. In Dynamics, the stress tensor is generalized in the form of the stress-mass tensor T while the motion of the continuum can be described by s' and $\partial s'/\partial X$ as explained at Section 9.1. Forgeting provisionaly the dependance with respect to s', we would like to specify the conditions of consistency of constitutive laws modeled by a map:

$$T = \mathcal{F}\left(\frac{\partial s'}{\partial X}\right) .$$

Now, we present the **principle of material indifference**. The key idea is to have a look to the Lagrangian representation.

Theorem 9.12 *In the Lagrangian representation, the stress-mass tensor T' depends on $\partial s'/\partial X$ through* ***right Cauchy strains*** *$C = F^T F$.*

Proof. Let X be any Galilean coordinate system and X' be the coordinate system (9.5) associated to the Lagrangian coordinates s'. According to Theorem 9.2, the Jacobian matrix of the coordinate change $X' \mapsto X$ is a deformation:

$$\frac{\partial X}{\partial X'} = \begin{pmatrix} 1 & 0 \\ v & F \end{pmatrix} \in \mathbb{GD} . \tag{9.60}$$

For another Galilean coordinate system $\bar{X}$, the the Jacobian matrix of the coordinate change $X' \mapsto \bar{X}$ is also deformation:

$$\frac{\partial \bar{X}}{\partial X'} = \begin{pmatrix} 1 & 0 \\ \bar{v} & \bar{F} \end{pmatrix} \in \mathbb{GD} .$$

Then, one has:

$$\frac{\partial \bar{X}}{\partial X'} = \frac{\partial \bar{X}}{\partial X}\frac{\partial X}{\partial X'} ,$$

where the Jacobian matrix of the coordinate change $X \mapsto \bar{X}$ is a linear Galilean tranformation:

$$\frac{\partial \bar{X}}{\partial X'} = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix} ,$$

that provides:

$$\bar{v} = u + R\,v, \qquad \bar{F} = R\,F \; . \tag{9.61}$$

As T' are the components of the stress-mass tensor $\mathbf{T}$ in the coordinate system X' associated to the Lagrangian coordinates s', it is preserved by any change $X \mapsto \bar{X}$ of Galilean coordinate system. Hence if there is some constitutive law:

$$T' = \mathcal{F}'\left(\frac{\partial s'}{\partial X}\right) ,$$

the value T' must depend on the variable $\partial s'/\partial X$ through a quantity:

$$C = \mathcal{C}\left(\frac{\partial s'}{\partial X}\right) ,$$

invariant by Galilean transformations. In other word, we claim that:

$$\left\{\mathcal{C}\left(\frac{\partial s'}{\partial X}\right) = \mathcal{C}\left(\frac{\partial s'}{\partial \bar{X}}\right)\right\} \qquad \Rightarrow \qquad \left\{\mathcal{F}'\left(\frac{\partial s'}{\partial X}\right) = \mathcal{F}'\left(\frac{\partial s'}{\partial \bar{X}}\right)\right\} .$$

There exists a unique map $\tilde{\mathcal{F}}' = \mathcal{F}'/\mathcal{C}$ from $\mathbb{M}_{33}^{symm}$ into $\mathbb{M}_{44}^{symm}$ such that:

$$\mathcal{F}' = \tilde{\mathcal{F}}'\,\mathcal{C} \; .$$

It reminds to guess the invariant C. As the variable $\partial s'/\partial X$ is obtained by erasing the first row in the inverse $\partial X'/\partial X$ of the Jacobian matrix (9.60):

$$\frac{\partial s'}{\partial X} = \begin{pmatrix} -F^{-1}v & F^{-1} \end{pmatrix} , \tag{9.62}$$

we wish to identify invariants of v and F, variables which are transformed under Galilean transformations according to (9.61). As that was done at Section 3.1.1 for the dynamical torsor, we intend annihilating some of them by suitable Galilean transformations (because zeros are constant then obvious invariants). Starting in any coordinate system X, we pick $u = -R^T v$ which annihilates $\bar{v}$. There is nothing more to do because the rotation R obviously cannot annihilate $\bar{F}$. We can verify that $C = F^T F$ are invariant since:

$$\bar{C} = \bar{F}^T \bar{F} = (R\,F)^T (R\,F) = F^T (R^T R)\,F = F^T F = C \; ,$$

that achieves the proof. ■

It is worth to notice that the matrix C of right Cauchy strains is symmetric and, owing to (9.2):

$$J = \det(F) = \sqrt{\det(C)} \; .$$

Recovering the variable s', owing to (9.43), (9.54) and according to Theorem 9.11, the constitutive law in the Lagrangian representation reads:

$$\rho' = \frac{\rho_0(s')}{\sqrt{\det(C)}}$$
$$\sigma' = \sigma'(s', C) \ . \tag{9.63}$$

If the continuum is **homogeneous**, *i.e.* this material properties are the same at each point, ρ' and σ' does not depends explicitly on s'. Otherwise, it is **heterogeneous**.

We can now come back to the Eulerian representation of the constitutive law thanks to the transformation laws (9.44) and (9.45) of the stress-mass tensor:

$$T = \begin{pmatrix} \frac{\rho_0(s')}{\sqrt{\det(C)}} & \frac{\rho_0(s')}{\sqrt{\det(C)}} \, v^T \\ \frac{\rho_0(s')}{\sqrt{\det(C)}} \, v & \frac{\rho_0(s')}{\sqrt{\det(C)}} \, v \, v^T - F \, \sigma'(s', C) \, F^T \end{pmatrix} .$$

As example of such constitutive laws, we can quote the **barotropic fluids** of which the material stresses read:

$$\sigma' = -q \, (s', \det(C)) \, C^{-1} \ , \tag{9.64}$$

where q is a given scalar valued function representing the **pressure**. The corresponding stress-mass tensor in Eulerian representation is:

$$T = \begin{pmatrix} \frac{\rho_0(s')}{\sqrt{\det(C)}} & \frac{\rho_0(s')}{\sqrt{\det(C)}} \, v^T \\ \frac{\rho_0(s')}{\sqrt{\det(C)}} \, v & \frac{\rho_0(s')}{\sqrt{}} \, v \, v^T + q \, (s', \det(C)) \, 1_{\mathbb{R}^3} \end{pmatrix} . \tag{9.65}$$

Owing to (13.13), the balance of linear momentum reads for barotropic fluids:

$$\boxed{\rho \, \frac{dv}{dt} = -grad \, q + f + \rho(g - 2 \, \Omega \times v) \ .} \tag{9.66}$$

If the time is fixed at a given date t, we can considered only **spatial tensors**. Arising the time, Galileo's group is reduced to the group of special Euclidean transformations preserving the components of the covariant metric tensor $\breve{\boldsymbol{G}}$. The material stresses σ'^{ij}, gathered in the matrix σ', are the components in the Lagrangian representation of a 2-contravariant tensor $\boldsymbol{\sigma}$. On the other hand, Theorem 9.3 showed that the position of the body at time t is given by a map:

$$\varphi_t : s' \mapsto r = \varphi_t(s') = \varphi(t, s') \ ,$$

which is the local expression in given charts of a map $\boldsymbol{\varphi}_t$. Its tangent map $\boldsymbol{F}$ is represented in the coordinates s' and r by the matrix F. To find the free coordinate version of the definition $C = F^T F$, let us remark that it reads in tensorial notations:

$$C_{ij} = \delta_{kl} F_i^k F_j^l \ .$$

Thus right Cauchy strains are the components of a 2-covariant spatial tensor $\boldsymbol{C}$ which is the pull-back by $\boldsymbol{\varphi}_t$ of the metric tensor $\breve{\boldsymbol{G}}$:

$$\boldsymbol{C} = \boldsymbol{\varphi}_t^* \breve{\boldsymbol{G}}$$

Omitting the dependance with respect to s', we can therefore write the constitutive law (9.63) in a coordinate-free form:

$$\boldsymbol{\sigma} = \boldsymbol{H}(\boldsymbol{C}) \ . \tag{9.67}$$

Having a look to the definition (8.2) of the strain tensor suggests to express as usual $\boldsymbol{\sigma}$ with respect to $\boldsymbol{C}$ through:

$$\boldsymbol{E} = \frac{1}{2}\left[\boldsymbol{\varphi}_t^* \breve{\boldsymbol{G}} - \breve{\boldsymbol{G}}\right] = \frac{1}{2}\left[\boldsymbol{C} - \breve{\boldsymbol{G}}\right] \ ,$$

called **Euler-Lagrange strain tensor**. In every Galilean coordinate system, Gram's matrix being the identity, it is represented by:

$$E = \frac{1}{2}\left(C - 1_{\mathbb{R}^3}\right) = \frac{1}{2}\left(F^T F - 1_{\mathbb{R}^3}\right) \ . \tag{9.68}$$

It gives a new version of the constitutive law (9.67):

$$\boldsymbol{\sigma} = \boldsymbol{H}(\boldsymbol{E}) \ . \tag{9.69}$$

9.7 Hyperelastic materials and barotropic fluids

Another example are the **hyperelastic materials** of which the behaviour is reversible. In Linear Elasticity (Section 8.3.1), we formulated the reversibility condition in terms of internal work by unit volume, that does not create difficulties because the strains are very small and the volume is almost unchanged. On the other hand, for fluids and solids exhibiting very large deformations, it is preferable to relate the internal power to the mass to model the constitutive law and to formulate the reversibility condition in term of **specific internal work**, that is of internal work by mass unit $\mathcal{P}_{int}/\rho$. Hence we claim that for any loop in the space of Euler-Lagrange strain tensors:

$$\oint Tr\left(\frac{\sigma'}{\rho}\, dE\right) = 0 \ , \tag{9.70}$$

where σ' is given by:

$$\sigma' = H'(E) \ ,$$

and ρ is also depending on E through C. Reasoning as in Subsection 3.3.2, we obtain the equivalent local condition:

$$\forall dE, \delta E, \qquad d\left(\frac{\sigma'}{\rho}\right) : \delta E - \delta\left(\frac{\sigma'}{\rho}\right) : dE = 0 \ ,$$

As in Section 8.3.1, there exists a **reversible energy potential** e_{int} generating the constitutive law in terms of material stresses:

$$\sigma' = \rho \, \frac{\partial e_{int}}{\partial E} \ . \tag{9.71}$$

Moreover, differentiating (9.68), one has $dC = 2\,dE$ and taking into account the definition (13.22) of the derivative with respect to a matrix, one has:

$$\sigma' = 2\,\rho \, \frac{\partial e_{int}}{\partial C} \ . \tag{9.72}$$

Combining (9.45) with (9.71) and (9.72) gives the constitutive law in terms of spatial stresses:

$$\sigma = \rho \, F \, \frac{\partial e_{int}}{\partial E} \, F^T = 2\rho \, F \, \frac{\partial e_{int}}{\partial C} \, F^T \ . \tag{9.73}$$

Let us show that barotropic fluids are particular cases of hyperelastic materials. Indeed, let us chose a potential e_{int} depending on C through $J = \det(F) = \sqrt{\det(C)}$. Applying the chain rule to calculate (9.72):

$$\sigma' = 2\,\rho \, \frac{de_{int}}{dJ} \, \frac{\partial J}{\partial(\det(C))} \, \frac{\partial}{\partial C} \, (\det(C)) \ .$$

Taking into account (9.54) and (13.24) leads to:

$$\sigma' = \rho_0 \, \frac{de_{int}}{dJ} \, C^{-1} \ ,$$

which can be identified to (9.64) provided:

$$q = -\rho_0 \, \frac{de_{int}}{dJ} \ . \tag{9.74}$$

Let us come back now to the general case of hyperelastic materials. First of all, let us remark that (9.55) reads:

$$\frac{d}{dt} \, (F^{-1}) = -F^{-1} \, \frac{\partial v}{\partial r} \ ,$$

that, owing to (13.25), leads to:

$$\frac{\partial v}{\partial r} = \frac{dF}{dt} \, F^{-1} \ . \tag{9.75}$$

and:

$$\frac{dF}{dt} = \frac{\partial v}{\partial r} \, F \ .$$

Hence, differentiating (9.68) gives:

$$\frac{dE}{dt} = \frac{1}{2} \, \frac{dC}{dt} = \frac{1}{2} \, \left(\frac{dF}{dt}^T \, F + F^T \frac{dF}{dt} \right) = \frac{1}{2} \, F^T \left[\frac{\partial v}{\partial r} + \left(\frac{\partial v}{\partial r} \right)^T \right] \, F \ ,$$

or, introducing the strain velocity (8.6):

$$\frac{dE}{dt} = \frac{1}{2}\,\frac{dC}{dt} = F^T D\, F \ . \tag{9.76}$$

Remenbering the dependence of e_{int} with respect to s' and differentiating it provides:

$$\rho\,\frac{de_{int}}{dt} = \rho\,\frac{\partial e_{int}}{\partial s'}\;\frac{ds'}{dt} + \rho\, Tr\,\left(\frac{\partial e_{int}}{\partial E}\,\frac{dE}{dt}\right)\ ,$$

or, owing to (9.11) and (9.76):

$$\rho\,\frac{de_{int}}{dt} = \rho\, Tr\,\left(\frac{\partial e_{int}}{\partial E}\, F^T D\, F\right) = Tr\,\left(\rho\, F\,\frac{\partial e_{int}}{\partial E}\, F^T D\right)\ ,$$

that is, taking into account the expression (9.73) of the spatial stresses, the internal power by volume unit (8.5):

$$\rho\,\frac{de_{int}}{dt} = Tr\ (\sigma D)\ \ , \tag{9.77}$$

We are now able to transform the general expression of the balance of energy (9.59) for the particular case of hyperelastic materials with zero volume forces f, that gives:

$$\rho\,\frac{d}{dt}\left(\frac{1}{2}\ \parallel v\parallel^2\right) = div\,(\sigma v) - Tr\,(\sigma D) + \rho g\cdot v\ .$$

Taking into account (9.77) and expressiong the gravity in terms of the Galilean gravitation potentials thanks to (6.13):

$$\rho\,\frac{d}{dt}\,\left(\frac{1}{2}\ \parallel v\parallel^2 + e_{int}\right) = div\,(\sigma v) - \rho\left(grad\,\phi + \frac{\partial A}{\partial t}\right)\cdot v\ ,$$

or:

$$\rho\,\frac{d}{dt}\,\left(\frac{1}{2}\ \parallel v\parallel^2 + \phi + e_{int}\right) = div\,(\sigma v) + \rho\left(\frac{\partial\phi}{\partial t} - \frac{\partial A}{\partial t}\cdot v\right)\ . \tag{9.78}$$

Inspired by the Hamiltonian (6.15) of a particle subjected to a Galilean gravitation and introducing the reversible energy potential, we define the **Hamiltonian density** as:

$$\mathcal{H} = \rho\,\left(\frac{1}{2}\parallel v\parallel^2 + \phi + e_{int}\right)\ . \tag{9.79}$$

or $\mathcal{H} = \rho\eta$ after introducing for sake of easiness the **specific Hamiltonian**:

$$\eta = \frac{1}{2}\parallel v\parallel^2 + \phi + e_{int}\ . \tag{9.80}$$

Taking into account the balance of mass (9.53), we have:

$$\frac{d\mathcal{H}}{dt} = \frac{d}{dt}\,(\rho\eta) = \rho\,\frac{d\eta}{dt} + \frac{d\rho}{dt}\,\eta = \rho\,\left(\frac{d\eta}{dt} - \eta\, div\, v\right)\ ,$$

which, owing to Definition (9.4 of the material derivative, gives:

$$\rho \frac{d\eta}{dt} = \frac{d\mathcal{H}}{dt} + \rho\eta \, div\, v = \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial r}\, v + \mathcal{H}\, div\, v \ ,$$

and, because of (12.38):

$$\rho \frac{d\eta}{dt} = \frac{\partial \mathcal{H}}{\partial t} + div\, (\mathcal{H} v) \ . \tag{9.81}$$

Introducing this expression into (9.78), we obtain a new version of the balance of energy dedicated to the hyperelastic materials and in particular to the barotropic fluids:

$$\boxed{\frac{\partial \mathcal{H}}{\partial t} + div\ \left(\mathcal{H} v - \sigma v\right) = \rho \left(\frac{\partial \phi}{\partial t} - \frac{\partial A}{\partial t} \cdot v \right) \ .} \tag{9.82}$$

For a time independent gravitation field, the right hand member vanishes.

Chapter 10

More about Calculus of Variations

10.1 Calculus of variation and tensors

Altough we have well progressed through the continuum mechanics, let us have a backward look to the variational formulation of the particle dynamics presented in Chapter 6 without apparent link with the tensors. Considering the 4-velocity $\vec{\boldsymbol{U}}$ of which the components in Galilean coordinate systems are:

$$U = \begin{pmatrix} 1 \\ v \end{pmatrix} ,$$

and the Galilean symmetric 2-covariant linear tensor $\boldsymbol{G}$ represented by the matrix (9.17):

$$G = \begin{pmatrix} -2\phi & A^T \\ A & 1_{\mathbb{R}^3} \end{pmatrix} ,$$

expressed in terms of potentials of the Galilean gravitation, it is easy to see that the Lagrangian (6.12) of a particle moving within the Galilean gravitation field can read using contracted products:

$$\mathcal{L} = \frac{m}{2} \vec{\boldsymbol{U}} \cdot \boldsymbol{G} \cdot \vec{\boldsymbol{U}} = \frac{m}{2} G_{\alpha\beta} U^\alpha U^\beta , \tag{10.1}$$

where:

$$G_{ij} = \delta_{ij}, \qquad G_{0i} = G_{i0} = A_i, \qquad G_{00} = -2\,\phi . \tag{10.2}$$

Euler-Lagrange equations (6.7) read in indicial notations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{v}^i}\right) - \frac{\partial \mathcal{L}}{\partial r^i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{U}^i}\right) - \frac{\partial \mathcal{L}}{\partial X^i} = 0 ,$$

where the Latin index i runs only from 1 to 3. Applying them to the Lagrangian (10.1), one has:

$$m \left(\frac{d}{dt}\left(G_{\alpha i} U^\alpha + G_{i\beta} U^\beta \right) - \frac{\partial G_{\alpha\beta}}{\partial X^i} U^\alpha U^\beta \right) = 0 .$$

Renaming the dummy indices in the former term and taking into account the symmetry of $\boldsymbol{G}$, one has:

$$m\left(2\,\frac{d}{dt}\left(G_{i\beta}U^{\beta}\right)-\frac{\partial G_{\alpha\beta}}{\partial X^{i}}U^{\alpha}U^{\beta}\right)=0\ .$$

Differentiating the first term gives:

$$m\left(G_{i\beta}\dot{U}^{\beta}+U^{\alpha}\frac{\partial G_{i\beta}}{\partial X^{\alpha}}U^{\beta}-\frac{1}{2}\,\frac{\partial G_{\alpha\beta}}{\partial X^{i}}U^{\alpha}U^{\beta}\right)=0\ .$$

Splitting the second term into two balanced ones and renaming dummy indices, we obtain:

$$m\left(G_{i\beta}\dot{U}^{\beta}+[\alpha\beta,i]\ U^{\alpha}U^{\beta}\right)=0\ . \tag{10.3}$$

introducing the symbols:

$$[\alpha\beta,\rho]=\frac{1}{2}\left(\frac{\partial G_{\rho\beta}}{\partial X^{\alpha}}+\frac{\partial G_{\rho\alpha}}{\partial X^{\beta}}-\frac{\partial G_{\alpha\beta}}{\partial X^{\rho}}\right)\ ,$$

which are obviously symmetric:

$$[\rho\alpha,\beta]=[\alpha\rho,\beta]\ .$$

Using $A_i=\delta_{ij}A^j$ and taking into account (10.2), the explicit calculation of these symbols gives:

$$[00,0]=-\frac{\partial\phi}{\partial t},\qquad [00,i]=\frac{\partial\phi}{\partial r^i}+\frac{\partial A_i}{\partial t},\qquad [0i,0]=-\frac{\partial\phi}{\partial r^i}\ , \tag{10.4}$$

$$[0i,j]=\frac{1}{2}\left(\frac{\partial A_j}{\partial r^i}-\frac{\partial A_i}{\partial r^j}\right),\qquad [ij,0]=\frac{1}{2}\left(\frac{\partial A_j}{\partial r^i}+\frac{\partial A_i}{\partial r^j}\right)\ . \tag{10.5}$$

Remarking that $U^0=1$, Euler-Lagrange equations in the form (10.3) is reduced to:

$$m\left(G_{ij}\dot{U}^j+[00,i]+[0j,i]\ U^j+[j0,i]\ U^j\right)=m\left(\delta_{ij}\dot{v}^j+\frac{\partial\phi}{\partial r^i}+\frac{\partial A_i}{\partial t}+\left(\frac{\partial A_j}{\partial r^i}-\frac{\partial A_i}{\partial r^j}\right)\,v^j\right)=0\,,$$

which reads in matrix notations:

$$m\left(\dot{v}+grad\,\phi+\frac{\partial A}{\partial t}+(curl\,A)\times v\right)=0$$

Taking into account (6.13), we recover once again the equation (3.46) of motion:

$$m\dot{v}=m\,(g-2\,\Omega\times v)\ .$$

10.2 Action principle for the dynamics of continua

Our goal now is deducing the other conservation equations, namely the balance of energy and linear momentum, from a variational principle. In the previous chapter, the motion of the continuum is modelized through a field defined on the space-time:

$$(t, r) \mapsto s' = \kappa(t, r) \ .$$

So the equation of the trajectory of the particle identified by s' is:

$$s' = \kappa(t, r) \ .$$

Its gradient $\partial s'/\partial X$ is given by (9.62), then the relevant variable for the constitutive law are:

- the velocity:

$$v = -F \, \frac{\partial s'}{\partial t} = -\frac{\partial r}{\partial s'} \, \frac{\partial s'}{\partial t} \ , \tag{10.6}$$

- the right Cauchy strains:

$$C = F^T F = \left(\frac{\partial r}{\partial s'} \right)^T \frac{\partial r}{\partial s'} \ . \tag{10.7}$$

This suggests to consider a Lagrangian of this form:

$$\mathcal{L} : \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 4} \longrightarrow \mathbb{R} : (X, s', z) \longmapsto \lambda = \mathcal{L}(t, s', z) \ .$$

and the corresponding action principle:

$$\alpha \ \left[s' \right] = \int_\Omega \mathcal{L} \ \left(X, s', \frac{\partial s'}{\partial X} \right) d^4 X$$

where the Lagrangian depends on the field s' and its first derivatives, defined on a bounded open subset Ω of the space-time $\mathbb{R}^4$. As we are only interested in what follows by the variational equations in the interior of Ω, we consider simple boundary conditions with the value of s' imposed on $\partial\Omega$.

In order to obtain the conservation identities, we use a special form of the calculus of variation [Comment 1]. The new viewpoint which consists in performing variations not only on the field s' and its derivatives but also on the variable X. To explicit them, we consider a new parametrization given by a regular mapping $X = \psi\ (Y)$ of class C^1 and perform the variation of the function ψ, the new variable being Y. After calculating the variation of the action, we will consider the particular case where the function ψ is the identity of Ω. Hence we start with

$$\alpha \ \left[X, s' \right] = \int_{\Omega'} \mathcal{L} \ \left(f(Y), s', \frac{\partial s'}{\partial Y} \, \frac{\partial Y}{\partial X} \right) \ \det \ \left(\frac{\partial X}{\partial Y} \right) d^4 Y$$

where $\Omega' = \psi^{-1}(\Omega)$ and the variables of the functional are now both X and s'. For sake of easiness, we introduce the 3-row

$$F = -\frac{\partial \mathcal{L}}{\partial s'}$$

the 4-row

$$H = \frac{\partial \mathcal{L}}{\partial X}$$

and the 4×3 matrix

$$P = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial s'}{\partial X}\right)}$$

The variation of the action reads

$$\delta\alpha = \int_{\Omega'} [\left(Tr\ \left(P\ \delta\ \left(\frac{\partial s'}{\partial Y}\ \frac{\partial Y}{\partial X}\right)\right) - F\ \delta s' + H\, \delta X\right)\ \det\ \left(\frac{\partial X}{\partial Y}\right)$$

$$+ \mathcal{L}\ \delta\ \left(\det\ \left(\frac{\partial X}{\partial Y}\right)\right]\, d^4Y \quad (10.8)$$

First of all, we calculate the variation of the derivative of the field in terms of the derivative of its variation

$$\delta\ \left(\frac{\partial s'}{\partial Y}\ \frac{\partial Y}{\partial X}\right) = \delta\left(\frac{\partial s'}{\partial Y}\right)\ \frac{\partial Y}{\partial X} + \frac{\partial s'}{\partial Y}\ \delta\ \left(\frac{\partial X}{\partial Y}\right)^{-1}$$

$$\delta\ \left(\frac{\partial s'}{\partial Y}\ \frac{\partial Y}{\partial X}\right) = \frac{\partial}{\partial Y}(\delta s')\ \frac{\partial Y}{\partial X} - \frac{\partial s'}{\partial Y}\ \frac{\partial Y}{\partial X}\ \frac{\partial}{\partial Y}(\delta X)\ \frac{\partial Y}{\partial X} \quad (10.9)$$

Incidently, it is worth noting that when $X = Y$

$$\delta\ \left(\frac{\partial s'}{\partial X}\right) = \frac{\partial}{\partial X}(\delta s') - \frac{\partial s'}{\partial X}\ \frac{\partial}{\partial X}(\delta X)$$

This formula shows that, unlike the usual rule used in the classical calculus of variation (see (6.4)), the derivative symbols $\partial/\partial X$ and δ may not be permuted in the present approach. Next, owing to (13.23), one has

$$\delta\ \left(\det\ \left(\frac{\partial X}{\partial Y}\right)\right) = Tr\ \left(\frac{\partial}{\partial Y}(\delta X)\ adj\left(\frac{\partial X}{\partial Y}\right)\right) \quad (10.10)$$

Introducing the expressions 10.9 and 10.10 into the variation of the action 10.8 gives

$$\delta\alpha = \int_{\Omega'} [Tr\ \left(P\ \frac{\partial}{\partial Y}(\delta s')\ \det\ \left(\frac{\partial X}{\partial Y}\right)\ \frac{\partial Y}{\partial X}\right) - \det\ \left(\frac{\partial X}{\partial Y}\right)\ (F\ \delta s' + H\, \delta X)$$

$$+ Tr\ \left(\mathcal{L}\ \frac{\partial}{\partial Y}(\delta X)\ adj\left(\frac{\partial X}{\partial Y}\right) - P\ \frac{\partial s'}{\partial X}\ \frac{\partial}{\partial Y}(\delta X)\ \det\ \left(\frac{\partial X}{\partial Y}\right)\ \frac{\partial Y}{\partial X}\right)]\ d^4Y \quad (10.11)$$

Taking into account (12.7) and introducing the 4×4 matrix:

$$T = P \, \frac{\partial s'}{\partial X} - \mathcal{L} \; 1_{\mathbb{R}^4} \tag{10.12}$$

the variation of the action 10.11 becomes

$$\delta\alpha = \int_{\Omega'} [\; Tr \, \left(adj \, \left(\frac{\partial X}{\partial Y} \right) \, P \, \frac{\partial}{\partial Y}(\delta s') \right) \; - \det \, \left(\frac{\partial X}{\partial Y} \right) \, (F \; \delta s' + H \, \delta X)$$

$$- \, Tr \, \left(adj \, \left(\frac{\partial X}{\partial Y} \right) \; T \; \frac{\partial}{\partial Y}(\delta X) \; \right)] \; d^4 Y \tag{10.13}$$

Taking into account (13.19), we integrate by part in 10.13. Taking into account the fact that the values of s' and X are imposed on the boundary, the surface integrals vanish and we obtain

$$\delta\alpha = \int_{\Omega'} (\, - \left[\; div_Y \left(adj \, \left(\frac{\partial X}{\partial Y} \right) \; P \right) + \det \, \left(\frac{\partial X}{\partial Y} \right) \; F \right] \delta s'$$

$$+ \left[div_Y \left(adj \, \left(\frac{\partial X}{\partial Y} \right) \; T \right) + \det \, \left(\frac{\partial X}{\partial Y} \right) H \right] \delta X) \; d^4 Y$$

where the index of *div* indicates with respect of which variable we differentiate. Finally, considering the particular case where $X = Y$, the variational principle reads

$$\delta\alpha = \int_{\Omega} (\, - [\; div_X P + F] \, \delta s' + [div_X T \; + H] \, \delta X) \; d^4 X = 0$$

The variation of s' and X being arbitrary, we obtain the equations of variation

$$div_X P + F = 0$$

$$div_X \, T + H = 0 \tag{10.14}$$

The first equation leads to a non linear partial derivative system which can be used to determine the unknown field s'. The last one gives extra conservation conditions [Comment 2]. In the next Section, we physically interprete it.

10.3 Explicit form of the variational equations

Now, we are interested in particular continuous media for which ones the Lagrangian depends on the first partial derivatives of s' through the velocity and configuration

$$\mathcal{L} \; \left(X, s', \frac{\partial s'}{\partial X} \right) = \mathcal{L} \; \left(s', v, C \right)$$

Its differential is

$$\delta \mathcal{L} = Tr \; \left(\frac{\partial \mathcal{L}}{\partial C} \; \delta C \right) + \frac{\partial \mathcal{L}}{\partial v} \; \delta v \tag{10.15}$$

Besides, differentiating the right Cauchy strains (10.7) and taking into account (13.25):

$$\delta C = \delta F^T F + F^T \delta F = -(F^T \delta (F^{-1})^T F^T F + F^T F\, \delta (F^{-1})\, F) \ ,$$

$$\delta C = -\left(F^T \delta \left(\frac{\partial s'}{\partial r}\right)^T C + C\, \delta \left(\frac{\partial s'}{\partial r}\right) F \right) \ ,$$

and differentiating the velocity (10.6):

$$\delta v = -\delta \left[\left(\frac{\partial s'}{\partial r}\right)^{-1} \right] \frac{\partial s'}{\partial t} - \frac{\partial r}{\partial s'}\, \delta \left(\frac{\partial s'}{\partial t}\right) = \frac{\partial r}{\partial s'}\, \delta \left(\frac{\partial s'}{\partial r}\right) \frac{\partial r}{\partial s'} \frac{\partial s'}{\partial t} - \frac{\partial r}{\partial s'}\, \delta \left(\frac{\partial s'}{\partial t}\right)$$

$$\delta v = F \left(\delta \left(\frac{\partial s'}{\partial r}\right) v + \delta \left(\frac{\partial s'}{\partial t}\right) \right)$$

next replacing both former expressions into 10.15, one has

$$\delta \mathcal{L} = Tr \left(\frac{\partial \mathcal{L}}{\partial C} F^T \delta \left(\frac{\partial s'}{\partial r}\right)^T C \right) + Tr \left(\frac{\partial \mathcal{L}}{\partial C} C\, \delta \left(\frac{\partial s'}{\partial r}\right)^T F \right)$$

$$- \frac{\partial \mathcal{L}}{\partial v} F\, \delta \left(\frac{\partial s'}{\partial r}\right) v - \frac{\partial \mathcal{L}}{\partial v} F\, \delta \left(\frac{\partial s'}{\partial t}\right) \tag{10.16}$$

By simple manipulations, taking into account the commutativity of the factors under the trace symbol and the symmetry of C, then of $\partial \mathcal{L} / \partial C$, the first two terms of the right hand side are equal to:

$$-Tr \left(\left(F \frac{\partial \mathcal{L}}{\partial C} C\, \delta \left(\frac{\partial s'}{\partial r}\right) \right) \right)$$

Besides, the third term reads:

$$-Tr \left(v \frac{\partial \mathcal{L}}{\partial v} F\, \delta \left(\frac{\partial s'}{\partial r}\right) \right)$$

Thus, expression 10.16 reads

$$\delta \mathcal{L} = -Tr \left(\left(2\, F \frac{\partial \mathcal{L}}{\partial C} C + v \frac{\partial \mathcal{L}}{\partial v} F \right) \delta \left(\frac{\partial s'}{\partial r}\right) \right)$$

$$- \frac{\partial \mathcal{L}}{\partial v} F\, \delta \left(\frac{\partial s'}{\partial t}\right) \tag{10.17}$$

On the other hand, introducing the 4-row

$$P_t = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial s'}{\partial t}\right)}$$

and the 3×3 matrix

$$P_r = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial s'}{\partial r}\right)}$$

we can express the differential of the Lagrangian as

$$\delta \mathcal{L} = P_t \ \delta \left(\frac{\partial s'}{\partial t}\right) + Tr \ \left(P_r \ \delta \left(\frac{\partial s'}{\partial r}\right)\right) \tag{10.18}$$

Comparing 10.17 and 10.18 leads to:

$$P_t = -\frac{\partial \mathcal{L}}{\partial v} F \ ,$$

$$P_r = -2\, F \frac{\partial \mathcal{L}}{\partial C} C - v \frac{\partial \mathcal{L}}{\partial v} F \ . \tag{10.19}$$

Inspired by the Lagrangian (6.12) of a particle subjected to a Galilean gravitation and introducing the reversible energy potential, we define the Lagrangian as:

$$\mathcal{L} = \frac{\rho}{2} \ \parallel v \parallel^2 + \rho\, A \cdot v - \rho\phi - W \ (s', \mathcal{C}) \ ,$$

$$\boxed{\mathcal{L} = \rho \left(\frac{1}{2} \ \parallel v \parallel^2 + A \cdot v - \phi - e_{int}(s', \mathcal{C})\right)} \tag{10.20}$$

where ρ is given by (9.54) and e_{int} is the specific internal energy. The case of the barotropic fluid can be easily obtained by considering that the reversible energy potential W depends on the right Cauchy strains through $\det C$. The expression 10.20 can be factorized as

$$\mathcal{L} \ (s', v, C) = \rho \ (s', C) \ L \ (s', v, C) \tag{10.21}$$

where ρ is defined by 9.56. Let us remark that applying (13.24) gives:

$$\frac{\partial \rho}{\partial C} = -\frac{\rho}{2} \ C^{-1}$$

Thus, differentiating 10.21, one has

$$\frac{\partial \mathcal{L}}{\partial C} \ = L \ \frac{\partial \rho}{\partial C} + \rho \ \frac{\partial L}{\partial C} = -\frac{\mathcal{L}}{2} C^{-1} + \rho \ \frac{\partial L}{\partial C}$$

Substituting this expression into 10.19, it holds:

$$P_r = -2\rho\, F \frac{\partial L}{\partial C} C + \left(\mathcal{L} \ 1_{\mathbb{R}^3} - v \ \frac{\partial \mathcal{L}}{\partial v}\right) \ F \tag{10.22}$$

Besides, 10.18 can be written in a more compact form

$$\delta\mathcal{L} = Tr\ \left(P\ \delta \left(\frac{\partial s'}{\partial X} \right) \right)$$

with

$$P = \begin{pmatrix} P_t \\ P_r \end{pmatrix}$$

Besides, we introduce the **generalized linear momentum** represented by the 3-column

$$\pi = grad_v \mathcal{L} = \rho\,(v + A) \tag{10.23}$$

and:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial v}\ v - \mathcal{L} = \rho\ \left(\frac{1}{2} \parallel v \parallel^2 + \phi + e_{int} \right)\ , \tag{10.24}$$

recovering the Hamiltonian density (9.79). Finally, the 4×4 matrix 10.12 can be decomposed by block

$$T = \begin{pmatrix} P_t \dfrac{\partial s'}{\partial t} - \mathcal{L} & P_t \dfrac{\partial s'}{\partial r} \\ \\ P_r \dfrac{\partial s'}{\partial t} & P_r \dfrac{\partial s'}{\partial r} - \mathcal{L} 1_3 \end{pmatrix}$$

Next, we calculate each block of T. For instance, owing to 10.22, one has

$$P_r \frac{\partial s'}{\partial r} - \mathcal{L} 1_3 = \sigma - v\ \frac{\partial \mathcal{L}}{\partial v}$$

with the symmetric 3×3 matrix

$$\sigma = -2\rho\, F\, \frac{\partial L}{\partial C}\, F^T = 2\rho\, F\, \frac{\partial e_{int}}{\partial C}\, F^T\ ,$$

identified to the spatial stresses, according to (9.73). Finally, the matrix is structured as [Comment 3]:

$$T = \begin{pmatrix} \mathcal{H} & -\pi^T \\ \mathcal{H} v - \sigma v & \sigma - v\,\pi^T \end{pmatrix} \tag{10.25}$$

10.4 Balance equations of the continuum

Next, we show that the variational equation 10.14 leads to the well known balance of energy and momentum

Theorem 10.1 *If T satisfied the equation of variation (10.14):*

$$div_X\, T + H = 0$$

then we have:

♢ *the balance of linear momentum:* $\rho \dfrac{dv}{dt} = (div\, \sigma)^T + \rho\; (g - 2\,\Omega \times v))$

♡ *the balance of energy:* $\dfrac{\partial \mathcal{H}}{\partial t} + div\;\; (\mathcal{H} v - \sigma v) = \rho \left(\dfrac{\partial \phi}{\partial t} - \dfrac{\partial A}{\partial t} \cdot v \right)$

Proof. ♢ **Balance of linear momentum.** Owing to (10.20), one has:

$$F = \left(\frac{\partial \mathcal{L}}{\partial t}, \frac{\partial \mathcal{L}}{\partial r} \right) = \left(\rho \left(v \cdot \frac{\partial A}{\partial t} - \frac{\partial \phi}{\partial t} \right), \rho \left(v^T \frac{\partial A}{\partial r} - \frac{\partial \phi}{\partial r} \right) \right) \tag{10.26}$$

We calculate the divergence of the matrix T using (13.11). Calculating the divergence of the last column of (10.25) and owing to 13.14, (10.14) gives:

$$-\frac{\partial}{\partial t}(\rho\,(v + A)^T) + div\,\sigma - div\,(\rho v)\,(v + A)^T - \rho v^T grad\,(v + A) + \frac{\partial \mathcal{L}}{\partial r} = 0 \tag{10.27}$$

or, expanding the first term:

$$-\rho\, \frac{\partial}{\partial t}(v + A)^T - \left(\frac{\partial \rho}{\partial t} + div\,(\rho v) \right)\, (v + A)^T - \rho v^T grad\,(v + A) + div\,\sigma + \frac{\partial \mathcal{L}}{\partial r} = 0$$

Taking into account (10.26) and Theorem 9.11, it holds:

$$-\rho\, \left(\frac{\partial v^T}{\partial t} + v^T grad\, v \right) + div\,\sigma - \rho\left(\frac{\partial \phi}{\partial r} + \frac{\partial A}{\partial t} \right) + \rho v^T \left(\frac{\partial A}{\partial r} - grad\, A \right) = 0$$

By transposition, one has, owing to (12.40) and (6.13):

$$-\rho \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial r}\, v \right) + div\,\sigma + \rho\; \left(g + 2 v^T j\,(\Omega) \right) = 0$$

that leads to the balance of linear momentum ♢

$$\rho \frac{dv}{dt} = (div\,\sigma)^T + \rho\; (g - 2\,\Omega \times v) \tag{10.28}$$

♡ **Balance of energy.** Calculating the divergence of the first column of (10.25), (10.14) gives:

$$\frac{\partial \mathcal{H}}{\partial t} + div\;\; (\mathcal{H} v - \sigma v) + \frac{\partial \mathcal{L}}{\partial r} = 0$$

Taking into account (10.26), one obtains the balance of energy:

$$\frac{\partial \mathcal{H}}{\partial t} + div\;\; (\mathcal{H} v - \sigma v) = \rho \left(\frac{\partial \phi}{\partial t} - \frac{\partial A}{\partial t} \cdot v \right)$$

that achieves the proof. ■

The attentive reader will observe that we recover –by a method enterely different from the one of the previous chapter– the balance of linear momentum (9.58) in absence of volume forces f and the balance of energy ((9.82)) for hyperelastic materials.

10.5 Comments for experts

[Comment 1] We perform a special form of the calculus of variation on the jet space of order one.

[Comment 2] This conservation equation is obtained in the spirit of Noether's theorem.

[Comment 3] Matrix T is the analogous in Galilean mechanics of the energy-momentum tensor in relativistic one.

Chapter 11

Thermodynamics of continua

11.1 Introduction

Before addressing the thermodynamics of continua, background ideas of thermodynamics are briefly recalled. Inspired by Carnot's works, Clausius showed in 1865 that the ratio Q_R/θ, where Q_R is the amount of heat absorbed in an isothermal and reversible process by a thermodynamic system at the absolute temperature θ, is a state function

$$\mathcal{S} = \frac{Q_R}{\theta} \tag{11.1}$$

which he calls the entropy. In 1877, Boltzmann threw a new light on this abstract physical quantity, by defining in statistical mechanics the entropy as proportional to the logarithm of the number of microscopic configurations that result in the macroscopic description of the system. To overcome the limitations of the previous approaches originally based on the study of thermal engines, Caratheodory proposed in 1908 an axiomatic approach. He considers reversible processes with varrying temperature. While the elementary heat supply δQ_R is not integrable, the elementary variation of entropy

$$d\mathcal{S} = \frac{1}{\theta}\,\delta Q_R$$

so is. Then the difference of entropy between two states of the system A and B, does not depends of the path to go from A to B

$$\mathcal{S}\,(B) - \mathcal{S}\,(A) = \int_A^B \frac{1}{\theta}\,\delta Q_R$$

The first law of the thermodynamics, formalized through the heat-friction experiments of Joule in 1843, claims that a thermodynamic system can store and hold energy but that its total energy is conserved

$$dE_{int} + dK = \delta W + \delta Q$$

where E_{int} is the internal energy of the system, K is the kinetic one, δW is the work done by surroundings and δQ is the element of heat absorbed by the system.

Reversible processes are ideal concepts but, in realistic situations, a part of the energy is lost or dissipated due to internal frictions producing heat. The second law of thermodynamics, originally stated by Clausius, claims that, for the reversible and irreversible processes, the total production of entropy is positive

$$\frac{\delta Q}{\theta} = \frac{\delta Q_R}{\theta} - \frac{\delta Q_I}{\theta} = d\mathcal{S} - \frac{\delta Q_I}{\theta} \geq 0$$

the equality being reached only for the reversible processes. In this formula, the sign before δQ_I is conventionally chosen because this irreversible heat is produced by the system itself.

The previous concepts were initially introduced to describe the behavior of systems, independantly of the mechanics of continua, but this two topics can be married in spirit of Truesdell's ideas ([39]) and his school. The basic idea is to apply the concepts of the thermodynamics to any volume element of a continuum to obtain local versions of the two principles consistent with Galileo's principle of relativity 1.13.

11.2 An extra dimension

In order to do not introduce early too much complexity, we begin with modelizing the thermodynamics in situations where the gravitation can be neglected. For a particle in uniform straight motion, classical integrals of the motion –mass, linear and angular momenta– were revealed as components of the dynamical torsor (Law 3.4) but there is an noticeable absent one, the kinetic energy:

$$e = \frac{1}{2} m \parallel v \parallel^2 \quad . \tag{11.2}$$

The difficulty to recover it is deep and overcoming it needs a strong change of viewpoint. The cornerstone idea is to add to the space-time an extra dimension roughtly speaking linked to the energy by a three step method that we shall be going to present in an heuristic way:

- As we are concerned with the uniform straight motion, we do not consider provisionally the gravitation. We start with a fictitious 5-dimensional affine space $\hat{\mathcal{U}}$ containing the space-time $\mathcal{U}$. We claim that any point $\hat{\boldsymbol{X}}$ of $\hat{\mathcal{U}}$ can be represented in some suitable coordinate systems by a column:

  $$\hat{X} = \begin{pmatrix} X \\ z \end{pmatrix} \in \mathbb{R}^5 ,$$

  in such way that the space-time is identified to the subspace $\hat{X}^4 = z = 0$ of $\hat{\mathcal{U}}$ and X gathers Galilean coordinates.

- We wish to build a group of affine transformations $d\hat{X}' \mapsto d\hat{X} = \hat{P}\, d\hat{X}' + \hat{C}$ of $\mathbb{R}^5$ such that, when acting on the space-time only, are Galilean ones. Clearly, the 5×5 matrix $\hat{P}$ is structured as:

$$\hat{P} = \begin{pmatrix} P & 0 \\ \Phi & \alpha \end{pmatrix} ,$$

where the 4-row Φ and the scalar α have to take an appropriate physical meaning.

- It is worth to notice that under a Galilean coordinate change $X' \longmapsto X$ characterized by a boost u and a rotation R, using the velocity addition formula 1.13, its transformation law is:

$$e = \frac{1}{2} m \parallel u + R\, v' \parallel^2 = \frac{1}{2} m \parallel u \parallel^2 + m\, u \cdot (R\, v') + \frac{1}{2} m \parallel v' \parallel^2 \quad .$$

Next we claim that the extra coordinate is linked to the energy as follows:

$$dz = \frac{e}{m}\, dt \; . \tag{11.3}$$

The division by m is guided by the fact that we wish the extra coordinate being universal, independent of the mass of particle moving in the space-time. According to $dt = dt'$ and $dr' = v' dt'$, we obtain:

$$dz = \frac{1}{2} \parallel u \parallel^2 dt' + u^T R\, dr' + dz' \; .$$

On this ground, we state:

Definition 11.1 *The* ***Bargmannian transformations*** *are affine transformations* $d\hat{X}' \mapsto d\hat{X} = \hat{P}\, d\hat{X}' + \hat{C}$ *of* $\mathbb{R}^5$ *such that.*

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 \\ u & R & 0 \\ \frac{1}{2} \parallel u \parallel^2 & u^T R & 1 \end{pmatrix} \; . \tag{11.4}$$

It is straightforward to verfy that the set of the Bargmannian transformations is a subgroup of $\mathbb{A}ff(5)$ called **Bargmann's group** and denoted $\mathbb{B}$ in the sequel. In particular, the inverse of (11.4) is:

$$\hat{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -R^T u & R^T & 0 \\ \frac{1}{2} \parallel u \parallel^2 & -u^T & 1 \end{pmatrix} \; . \tag{11.5}$$

Definition 11.2 *The coordinate systems of* $\hat{\mathcal{U}}$*, which are deduced one from the other by Bargmanian transformations are called* ***Bargmannian coordinate systems****.*

In Theorem 1.11, we define Galilean transformations as preserving some objects –uniform straight motions, durations, distances and angles, oriented volumes– but what Bargmannian transformations preserve? Combining (11.2) and (11.3) leads to:

$$\parallel dr \parallel^2 - 2\, dz\, dt = 0 \ ,$$

in every Bargmannian coordinate system. The left hand member is a quadratic form in $d\hat{X}$ then represents a symmetric 2-covariant tensor $\hat{\mathbf{G}}$ of which Gram's matrix in Bargmannian coordinate systems is:

$$\hat{G} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1_{\mathbb{R}^3} & 0 \\ 1 & 0 & 0 \end{pmatrix} \ .$$

As it is nondegenerate, $\hat{\mathbf{G}}$ is a covariant metric tensor. It is easy to verify that:

$$\hat{G}^2 = 1_{\mathbb{R}^5} \ ,$$

that proves the contravariant metric tensor $\hat{\mathbf{G}}^{-1}$ is represented by the same matrix as the covariant one:

$$\hat{G}^{-1} = \hat{G} \ .$$

11.3 Temperature vector and friction tensor

As subgroup of the affine group $\mathbb{A}ff(5)$, Bargmann's group naturally acts onto the tensors by restriction of their transformation laws. The $\mathbb{B}$-tensors are called **Bargmannian tensors**. To begin with, let us study the Bargmannian vectors $\hat{\vec{\boldsymbol{W}}}$ represented by a 5-column

$$\hat{W} = \begin{pmatrix} W \\ \zeta \end{pmatrix} = \begin{pmatrix} \beta \\ w \\ \zeta \end{pmatrix} \ , \tag{11.6}$$

where $W \in \mathbb{R}^4$, $w \in \mathbb{R}^3$ and $\beta, \zeta \in \mathbb{R}$. According to (11.5), their transformation law (12.21) gives:

$$\beta' = \beta \ , \qquad w' = R^T(w - \beta u), \qquad \zeta' = \zeta - w \cdot u + \frac{\beta}{2} \parallel u \parallel^2 \ .$$

Bargmannian transformations leave β invariant and there is no trouble to put β instead of β' in the sequel. By a method similar to the one of Section 3.1.1, we search the other invariants of $\hat{\vec{\boldsymbol{W}}}$ in the case that β does not vanish. Starting in any Bargmannian coordinate system $\hat{X}$ and we choose the Galilean boost:

$$u = \frac{w}{\beta} \ ,$$

which annihilates w' and reduces the last component to the generally non vanishing expression:

$$\zeta' = \zeta - w \cdot u - \frac{1}{2\,\beta} \parallel w \parallel^2 \,,$$

Conversely, let us consider a Bargmannian coordinate system $\hat{X}'$ in which the vector $\hat{\vec{\boldsymbol{W}}}$ has a **reduced form**:

$$\hat{W}' = \begin{pmatrix} \beta \\ 0 \\ \zeta_{int} \end{pmatrix} \,.$$

In the spirit of the boost method initiated in Section 3.1.2, we claim now the considered elementary volume is at rest. Let X be another Galilean coordinate system obtained from X' through a Galilean boost v. Applying the inverse transformation law of (11.5) with a Galilean boost v, we obtain:

$$\hat{W} = \begin{pmatrix} \beta \\ \beta\, v \\ \zeta_{int} + \frac{\beta}{2} \parallel v \parallel^2 \end{pmatrix} \,.$$

As $w = \beta\, v$, the last components becomes

$$\zeta = \zeta_{int} + \frac{1}{2\beta} \parallel w \parallel^2 \tag{11.7}$$

It is worth noting that under Bargmannian transformations, there is a unique invariant generator, the first component β. It is independent of the coordinate system and, for reasons that will appear latter, we claim that:

Definition 11.3 *when $\beta = 1\,/\,\theta$ is the* ***reciprocal temperature****, $\hat{\vec{\boldsymbol{W}}}$ is called the* ***temperature vector****.*

Let us observe also that:

$$W = \beta U \,.$$

The temperature 4-vector W is decomposed by block as:

$$W = \begin{pmatrix} \beta \\ w \end{pmatrix}$$

Definition 11.4 *The* ***friction tensor*** *is the 1-covariant and 1-contravariant mixed tensor:*

$$\mathbf{f} = \nabla\, \boldsymbol{W}$$

As the gravitation is provisionally neglected and is represented by Christoffel's symbols $\Gamma^{\alpha}_{\mu\beta}$, we assume for the moment these latter ones vanish. Taking into account(13.29), it is represented in a Galilean coordinate system by the 4×4 matrix

$$f = \nabla W = \frac{\partial W}{\partial X} = \begin{pmatrix} \frac{\partial \beta}{\partial t} & \frac{\partial \beta}{\partial r} \\ \frac{\partial\, w}{\partial t} & \frac{\partial\, w}{\partial r} \end{pmatrix} . \tag{11.8}$$

11.4 Momentum tensors and first principle

Definition 11.5 *A* ***momentum tensor*** *is a* 1*-covariant tensor* $\hat{\boldsymbol{T}}$ *on the* 5*-dimensional space* $\hat{\mathcal{U}}$ *with vector values in the space-time* $\mathcal{U}$ *such that it is represented in a bargmannian coordinate system by a* 4×5 *matrix structured as follows:*

$$\boxed{\hat{T} = \begin{pmatrix} \mathcal{H} & -p^T & \rho \\ k & \sigma_* & p \end{pmatrix} ,} \tag{11.9}$$

where $\mathcal{H} \in \mathbb{R}$, $p, k \in \mathbb{R}^3$ *and* $\sigma_* \in \mathbb{M}^{symm}_{33}$.

In indicial notation, the components of $\hat{\boldsymbol{T}}$ are:

$$\hat{T}^0_0 = \mathcal{H}, \qquad \hat{T}^0_i = -\delta_{ik} p^k, \qquad \hat{T}^0_4 = \rho \ ,$$
$$\hat{T}^j_0 = k^j, \qquad \hat{T}^j_i = \sigma^j_{*i}, \qquad \hat{T}^j_4 = p^j \ .$$

According to (13.4), the transformation law of $\hat{\boldsymbol{T}}$ is:

$$\boxed{\hat{T}' = P^{-1}\ \hat{T}\ \hat{P} \ ,} \tag{11.10}$$

itemizes in the already known relations (9.35), (9.36) and (9.37):

$$\rho' = \rho \ , \tag{11.11}$$

$$p' = R^T\,(p - \rho u) \ , \tag{11.12}$$

$$\sigma'_\star = R^T\,(\sigma_\star + u\,p^T + p\,u^T - \rho\,u\,u^T)\,R \ , \tag{11.13}$$

completed by two extra rules:

$$\boxed{\mathcal{H}' = \mathcal{H} - u \cdot p + \frac{\rho}{2} \parallel u \parallel^2 \ ,} \tag{11.14}$$

$$\boxed{k' = R^T \left(k - \mathcal{H}' u + \sigma_* u + \frac{1}{2} \parallel u \parallel^2 p\right) .} \tag{11.15}$$

It is worth noting that the hypothesis of symmetry of $\sigma_\star$ is consistent with the rule (11.13). The components ρ, p and $\sigma_\star$ can be physically identified with the mass density, the linear momentum and the dynamical stresses. To interpret the other components, we intend to annihilate components of $\hat{\boldsymbol{T}}$. As usual, we discuss only the case of non zero mass density. Starting in any Bargmannian coordinate system $\hat{X}$, we choose the Galilean boost:

$$u = \frac{p}{\rho} ,$$

which annihilates p', reduces (11.13) to (9.38) and transforms (11.14) and (11.15) as follows:

$$\mathcal{H}' = \mathcal{H} - \frac{1}{2\rho} \parallel p \parallel^2 ,$$

$$k' = R^T \left(k - \mathcal{H}\frac{p}{\rho} + \sigma_\star \frac{p}{\rho} \right) .$$

The components $\mathcal{H}'$ and k' obviously cannot be annihilated by a convenient choice of a rotation R. At the very most, we could diagonalize the symmetric matrix $\sigma_\star$ but it not useful now. Next, we use the boost method of Section 3.1.2. Let us consider a Bargmannian coordinate system in which the tensor field $\hat{\boldsymbol{T}}$ at a given point of coordinates $\hat{X}'$ has a **reduced form**:

$$\hat{T} = \begin{pmatrix} \rho e_{int} & 0 & \rho \\ h' & \sigma' & 0 \end{pmatrix} ,$$

for an elementary volume around the point r' at rest at time t'. Let $\hat{X}$ be another Bargmannian coordinate system obtained from $\hat{X}'$ through a Galilean boost v combined with a rotation R. Applying the inverse transformation law of (11.10):

$$\hat{T} = P \, \hat{T}' \hat{P}^{-1} , \tag{11.16}$$

we obtain:

$$p = \rho v, \qquad \sigma_\star = \sigma - \rho v v^T ,$$

$$\mathcal{H} = \rho \left(\frac{1}{2} \parallel v \parallel^2 + e_{int} \right) , \tag{11.17}$$

$$k = h + \mathcal{H} v - \sigma v , \tag{11.18}$$

according to the transformation law (9.40) of the spatial stresses σ and provided:

$$h = R \, h' . \tag{11.19}$$

The boost method turns out the physical meaning of the components:

- the quantities already identified in Section 9.4.2, the mass density ρ and the dynamic stresses $\sigma_\star$,
- the Hamiltonian density $\mathcal{H}$ defined by (9.79), apart from the potential ϕ (the gravitation being considered only latter),
- and the **energy flux** k composed of h –further identified to the **heat flux**–, the **hamiltonian flux** $\mathcal{H}v$ and the **stress flux** σv.

We could name $\hat{\boldsymbol{T}}$ the stress-mass-energy-momentum tensor but for briefness we call it momentum tensor. Finally, it has the form

$$\hat{T} = \begin{pmatrix} \mathcal{H} & -p^T & \rho \\ h + \mathcal{H}v - \sigma v & \sigma - vp^T & \rho v \end{pmatrix} \tag{11.20}$$

In the last column of (11.20), we spot the 4-**flux of mass** :

$$N = \rho\, U \tag{11.21}$$

Therefore, we can write

$$\hat{T} = \begin{pmatrix} T & N \end{pmatrix} \tag{11.22}$$

with

$$T = \begin{pmatrix} \mathcal{H} & -p^T \\ h + \mathcal{H}v - \sigma v & \sigma - vp^T \end{pmatrix}$$

In fact, it is more convenient to express the momentum tensor as (11.20), accounting for the following proposition.

Theorem 11.6 *The expression (11.20) of the momentum tensor is standard provided σ and h are changing according respectively to the rules (9.40) and (11.19).*

Proof Matrix (11.20) can be recasted as

$$\hat{T} = \begin{pmatrix} \mathcal{H} & -p^T & \rho \\ & & \\ h + \mathcal{H}\,\dfrac{p}{\rho} - \sigma\,\dfrac{p}{\rho} & \sigma - \dfrac{1}{\rho}\,p\,p^T & p \end{pmatrix} \tag{11.23}$$

Owing to (11.11), (9.40) and (11.12), one has

$$\sigma'_* = \sigma' - \frac{1}{\rho'}\,p'\,p'^T = R^T \sigma\, R^T - \frac{1}{\rho}\,R^T (p - \rho u)(p^T - \rho u^T)\,R\ ,$$

and developping:

$$\sigma'_\star = R^T\,(\sigma - \frac{1}{\rho}\,p\,p^T + u\,p^T + p\,u^T - \rho\,u\,u^T)\,R\ ,$$

which, owing to (9.39), is nothing else the tensorial rule (11.13).

In a similar way, taking into account (11.11), (11.19), (9.40) and (11.12), it holds:

$$k' = h' + \mathcal{H}' \, \frac{p'}{\rho'} - \sigma' \, \frac{p'}{\rho'} = R^T h + \frac{\mathcal{H}'}{\rho} \, R^T (p - \rho u) - \frac{1}{\rho} \, R^T \sigma \, (p - \rho u) \ .$$

Taking into account (11.14) gives:

$$k' = R^T h + \left(\mathcal{H} - u \cdot p + \frac{\rho}{2} \parallel u \parallel^2 \right) \, R^T \frac{p}{\rho} + \mathcal{H}' R^T u - R^T \sigma \, \frac{p}{\rho} + R^T \sigma \, u \ ,$$

and with some arrangements:

$$k' = R^T \left[h + \mathcal{H} \frac{p}{\rho} - \sigma \frac{p}{\rho} - \mathcal{H}' u + \left(\sigma - \frac{1}{\rho} p \, p^T \right) \, u + \frac{1}{2} \parallel u \parallel^2 p \right] \ ,$$

which, owing to (11.18) and (9.39), is nothing else the tensorial rule (11.15). ■

The advantage of the standard form (11.20) is that transformation laws (9.40) and (11.19) for σ and h are easier to manipulate that the corresponding transformation laws (11.13) and (11.15) for $\sigma_\star$ and k.

Also, introducing:

$$\Pi = \left(\begin{array}{cc} \mathcal{H} & -p^T \end{array} \right) \ , \tag{11.24}$$

it is worth noting that the momentum (11.20) can be recast as:

$$T = U \, \Pi + \left(\begin{array}{cc} 0 & 0 \\ h - \sigma v & \sigma \end{array} \right) \ . \tag{11.25}$$

Owing to (11.8), (11.25) and the symmetry of σ leads to:

$$Tr \ (T \ f) = \Pi \, \frac{\partial W}{\partial X} \ U + Tr \ \left(\sigma \ \left(grad_s \ w - \frac{1}{2} \left(v \, \frac{\partial \beta}{\partial r} + grad \ \beta \ v^T \right) \right) \right) + h \cdot grad \ \beta \ . \tag{11.26}$$

The first principle of thermodynamics claims that the total energy of a system is conserved. We are now able to proposed an enhanced local version including the balance of mass and the equation of the motion (balance of linear momentum). It is based on the following result.

Theorem 11.7 *If $\hat{T}$ is divergence free:*

$$div_X \, \hat{T} = 0 \ ,$$

then, we have:

♢ ***balance of mass:*** $\dfrac{\partial \rho}{\partial t} + div \, (\rho \, v) = 0 \ ,$

♡ ***balance of linear momentum:*** $\rho \ \left[\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial r} \, v \right] = (div \ \sigma)^T \ ,$

♠ ***balance of energy:*** $\dfrac{\partial \mathcal{H}}{\partial t} + div\ \ (h + \mathcal{H} v - \sigma v) = 0\ .$

Proof. To calculate the divergence of the 4×5 matrix $\hat{T}$, we use (13.18) and some simple transformations already done to establish Euler's equations of motion (Law 9.9) and that will not be repeted here. ∎

Leaving provisionnally the volume force f and the gravitation aside to compare, it can be observed that these three balance equations were previously found but by distinc ways:

- In Section 9.5, we obtained directly the two former conditions starting from the study of the dynamical torsor of a 3D medium. The balance of energy was further deduced from them.
- In Section 10.4, the balance of mass was *a priori* assumed and the two latter equations were recovered by a variational principle.

In the thermodynamic framework, the three balance equations are together obtained thanks to the extra fifth dimension. On this ground and envolving the gravitation, we state the **first principle of the thermodynamics**:

Principle 11.8 *The momentum tensor of a continuum is covariant divergence free:*

$$\boxed{\boldsymbol{Div}\, \hat{\boldsymbol{T}} = \boldsymbol{0}\ .} \tag{11.27}$$

The covariant form of equation makes it consistent with Galileo's principle of relativity 1.13. The principle is general in the sense that it is valid for both reversible and dissipative continua. We are going now to describe successively these two kinds of media.

11.5 Reversible processes and thermodynamical potentials

To model the reversible processes, we need a new hypothesis inspired from the concept of potential as developped in Section 9.7, claiming that:

- These phenomena can be represented thanks to a scalar function ζ of the particle s', its first partial derivative $\partial s' / \partial X$ and the temperature vector W, called **Plank's potential**.
- According to the principle of material indifference 9.12, ζ depends on $\partial s' / \partial X$ through right Cauchy strains $C = F^T F$.

On this ground, we prove the following proposition.

Theorem 11.9 *If ζ is a smooth function of s', C and W, then there exists a momentum tensor $\hat{\boldsymbol{T}}_R$ defined by*

$$T_R = U \; \Pi_R + \begin{pmatrix} 0 & 0 \\ -\sigma_R v & \sigma_R \end{pmatrix} \tag{11.28}$$

with

$$\Pi_R = -\rho \, \frac{\partial \zeta}{\partial W} \tag{11.29}$$

$$\sigma_R = \frac{2\rho}{\beta} \, F \, \frac{\partial \zeta}{\partial C} \, F^T \; . \tag{11.30}$$

such that

- ♢ $Tr\left(\hat{T}_R \nabla \hat{W}\right) = 0$
- ♡ $T_R \; U = -\rho \; \left(\frac{\partial \zeta}{\partial W} \; U\right) \; U$
- ♠ $\hat{T}_R = \begin{pmatrix} T_R & N \end{pmatrix}$ *represents a momentum tensor* $\hat{\boldsymbol{T}}_R$
- ♣ $\hat{T}_R \; \hat{W} = \left(\zeta - \frac{\partial \zeta}{\partial W} \; W\right) \; N$

Proof. Taking into account (11.6), (11.8) and (11.22), condition ♢ reads:

$$(\nabla \zeta) \, N = -Tr \; (T_R f) \; , \tag{11.31}$$

or, in absence of gravitation:

$$\frac{\partial \zeta}{\partial X} \; N = -Tr \; (T_R f) \tag{11.32}$$

On the other hand, owing to (11.21), one has:

$$\frac{\partial \zeta}{\partial X} \; N = \rho \; (\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial r} \; v) = \rho \frac{d\zeta}{dt} \; .$$

As ζ depends on X through s', C and W, one has:

$$\frac{\partial \zeta}{\partial X} \; N = \rho \; \left(\frac{\partial \zeta}{\partial s'} \, \frac{ds'}{dt} + \frac{\partial \zeta}{\partial W} \, \frac{dW}{dt} + Tr \; \left(\frac{\partial \zeta}{\partial C} \, \frac{dC}{dt}\right)\right) \; . \tag{11.33}$$

Owing to (9.11), the first term of the right hand side vanishes. Taking into account (11.29), the second term is:

$$\rho \, \frac{\partial \zeta}{\partial W} \, \frac{dW}{dt} = -\Pi_R \, \frac{\partial W}{\partial X} \; U \; . \tag{11.34}$$

Next, we have to transform the last term of (11.33). Because of (9.76) and (11.30), one has:

$$Tr\left(\frac{\partial\zeta}{\partial C}\,\frac{dC}{dt}\right) = Tr\left(2\rho F\,\frac{\partial\zeta}{\partial C}\,F^T D\right) = Tr\,\left(\sigma_R\beta D\right) = Tr\,\left(\sigma_R\beta grad_s\,v\right)$$

that, taking into account (12.35), leads to:

$$\rho\,Tr\left(\frac{\partial\zeta}{\partial C}\,\frac{dC}{dt}\right) = -Tr\,\left(\sigma_R\,\left(grad_s\;w - \frac{1}{2}\left(v\,\frac{\partial\beta}{\partial r} + grad\;\beta\,v^T\right)\right)\right)\;. \quad (11.35)$$

Introducing the expressions (11.34) and (11.35) into (11.33) gives (11.32) and proves ♢, owing to (11.26). Moreover, owing to (11.28) and (11.29), it holds

$$T_R\;U = U\;\Pi_R\;U + \begin{pmatrix} 0 & 0 \\ -\sigma_R v & \sigma_R \end{pmatrix}\begin{pmatrix} 1 \\ v \end{pmatrix} = (\Pi_R\;U)\;\;U = -\rho\;\left(\frac{\partial\zeta}{\partial W}\;U\right)\;U$$

that proves ♡. Statement ♠ results of the fact that (11.28) has the standard form (11.25). Consequently, taking into account (11.6) and (11.22), one has

$$S = T_R\;W + \zeta\;N = \rho\;\left(\zeta - \beta\frac{\partial\zeta}{\partial W}\;U\right)\;U = \left(\zeta - \frac{\partial\zeta}{\partial W}\;W\right)\;N$$

and ♣ is satisfied. ■

Planck's potential ζ is a prototype of scalar functions called **thermodynamical potentials** and derived as follows:

- Comparing (11.25), (11.28) and (11.29) allows writing $\Pi_R = \begin{pmatrix} \mathcal{H}_R & -p^T \end{pmatrix}$ with:

$$\mathcal{H}_R = -\rho\;\frac{\partial\zeta}{\partial\beta}, \qquad p = \rho\;grad_w\,\zeta\;. \quad (11.36)$$

  Taking into account (11.7), it holds:

$$\mathcal{H}_R = -\rho\;\frac{\partial\zeta_{int}}{\partial\beta} + \frac{\rho}{2\beta^2}\;\|\;w\;\|^2\;,$$

  which allows recovering (11.17) because $w = \beta v$ and provided:

$$\boxed{e_{int} = -\frac{\partial\zeta_{int}}{\partial\beta}\;.} \quad (11.37)$$

  This potential, called **internal energy** (by unit volume), is function of s', C and W as derivative of ζ_{int}.

- According to Theorem 9.6, the 4-vector $\vec{\boldsymbol{S}} = \hat{\mathbf{T}}_R \, \hat{\vec{\boldsymbol{W}}}$ is a 4-flux that reads for convenience:

$$\vec{\boldsymbol{S}} = \rho \, s \, \vec{\boldsymbol{U}} = s \, \vec{\boldsymbol{N}}$$

represented by a 4-column:

$$S = \hat{T}_R \, \hat{W} \; . \qquad (11.38)$$

Then, setting $S = s\,N$ and taking into account ♣ of Theorem 11.9, one has:

$$s = \zeta - \frac{\partial \zeta}{\partial \beta}\,\beta - \frac{\partial \zeta}{\partial w}\,w = \zeta_{int} + \frac{1}{2\beta} \parallel w \parallel^2 - \left(\frac{\partial \zeta_{int}}{\partial \beta} - \frac{1}{2\beta^2} \parallel w \parallel^2 \right) \beta - \frac{1}{\beta} w^T w \; ,$$

that leads to:

$$\boxed{s = \zeta_{int} - \beta \, \frac{\partial \zeta_{int}}{\partial \beta}}$$

This quantity is called **specific entropy** and $\vec{\boldsymbol{S}}$ is its 4-flux. Hence, $-s$ appears as Legendre's transform (12.37) of ζ_{int} with respect to β. The latter equation and (11.37) are called **state equations** of the continuum.

- Moreover, we introduce a new potential called **Helmholtz free energy** (by unit volume):

$$\boxed{\psi = -\frac{1}{\beta} \, \zeta_{int} = -\theta \, \zeta_{int} \; .} \qquad (11.39)$$

By simple calculations, we obtain:

$$-\, e_{int} = \theta \, \frac{\partial \psi}{\partial \theta} - \psi \qquad (11.40)$$

$$-s = \frac{\partial \psi}{\partial \theta}$$

Hence, $-e_{int}$ appears as Legendre's transform of the function $\psi\,(s', C, \theta)$ with respect to θ. It is a function of s', C and $-s$ such that

$$\theta = \frac{\partial e_{int}}{\partial s}$$

Finally, we can find a nice integral of the motion:

Theorem 11.10 *For reversible processes, the 4-flux S is divergence free and the specific entropy s is an integral of the motion.*

Proof. Taking into account (11.38) and (13.19), it holds:

$$div\, S = (div\, \hat{T}_R)\, \hat{W} + Tr\, \left(\hat{T}_R \frac{\partial\, \hat{W}}{\partial X} \right) \; .$$

The momentum tensor $\hat{T}_R$ satisfies condition $\diamondsuit$ of Theorem 11.9 and, according to the first principle (Law 11.27), it is divergence free. Then the 4-flux of specific entropy S vanishes and one has, owing to (12.38):

$$div\, S = div\, (s\, N) = \frac{\partial s}{\partial X}\, N + s\, div\, N = 0 \; .$$

But, as seen in Theorem 11.7, the freeness of the divergence implies the balance of mass. As discussed at Section 9.5, this condition means the flux of mass N is divergence free and the last term of the previous equation vanishes. Then, because of the definition 9.4 of the material derivative, one has:

$$div\, S = \frac{\partial s}{\partial X}\, N = \rho\, \frac{\partial s}{\partial X}\, U = \rho\, \frac{ds}{dt} = 0 \; ,$$

that achieves the proof. ■

11.6 Dissipative continuum and heat transfer equation

In Section 11.4, we showed that the thermodynamical behaviour of a continuum is modeled by the momentum tensor $\hat{\mathbf{T}}$. By Theorem 11.6 we prove that $\hat{\mathbf{T}}_R$ is a momentum tensor. We define $\hat{\mathbf{T}}_I = \hat{\mathbf{T}} - \hat{\mathbf{T}}_R$, that amounts to introduce an additive decomposition of the momentum tensor

$$\hat{\mathbf{T}} = \hat{\mathbf{T}}_R + \hat{\mathbf{T}}_I \tag{11.41}$$

into a reversible part $\hat{\mathbf{T}}_R$ defined by theorem 11.9 and an irreversible part $\hat{\mathbf{T}}_I$ of which we will examine now the detailed representations. Owing to (11.36) , the momentum tensor $\hat{\mathbf{T}}_R$ given by (11.28) is represented by

$$\hat{T}_R = \left(\begin{array}{ccc} \mathcal{H}_R & -p^T & \rho \\ \mathcal{H}_R v - \sigma_R v & \sigma_R - v p^T & \rho v \end{array} \right)$$

Substracting the previous matrix to (10.25) leads to

$$\hat{T}_I = \left(\begin{array}{ccc} \mathcal{H}_I & 0 & 0 \\ h + \mathcal{H}_I v - \sigma_I v & \sigma_I & 0 \end{array} \right) \tag{11.42}$$

where:

- the hamiltonian density $\mathcal{H}_I = \mathcal{H} - \mathcal{H}_R$ is the opposite of the **irreversible heat source** (by unit volume),

- The column h is the **heat flux**.

- The symmetric 3×3 matrix $\sigma_I = \sigma - \sigma_R$ represents dissipative stresses (for instance due to viscous effects).

The tensorial rule (11.14)and (9.40) give:

$$\mathcal{H}'_I = \mathcal{H}_I \tag{11.43}$$

$$\sigma'_I = R\ \sigma_I\ R^T \tag{11.44}$$

Introducing also for convenience the **specific irreversible heat source** q_I such that

$$\mathcal{H}_I = -\rho q_I \tag{11.45}$$

we have $\mathcal{H} = \rho\,\eta$ where the definition (9.80) of the specific Hamiltonian must be modified for the additive decomposition (11.41) and no gravitation ($\phi = 0$):

$$\eta = \frac{1}{2} \parallel v \parallel^2 + e_{int} - q_I\ .$$

Taking into account (9.81) and (13.12), the balance energy (♠ of Theorem 11.7) reads:

$$\rho\,\frac{d\eta}{dt} + div\,h - (div\,\sigma)\,v - Tr\,\left(\frac{\partial v}{\partial r}\right) = 0\ ,$$

Taking into account the symmetry of σ, the definition (8.6) of the strain velocity D and the balance of linear momentum (♡ of Theorem 11.7), the balance of energy becomes:

$$\rho\,\frac{de_{int}}{dt} = Tr\ (\sigma\,D) - div\ h + \rho\,\frac{dq_I}{dt}\ . \tag{11.46}$$

With respect to the corresponding formula (9.77) for the reversible processes, the dissipative case exhibits the two extra terms due to the heat flux and the irreversible energy source.

A cornerstone consequence of is the heat transfer equation. By differentiation of (11.40) and well known calculations, we have:

$$\rho\,\frac{de_{int}}{dt} = \rho\,\left(\frac{d\psi}{dt} - \frac{d\theta}{dt}\frac{\partial\psi}{\partial\theta} - \theta\,\frac{d}{dt}\,\left(\frac{\partial\psi}{\partial\theta}\right)\right)\ .$$

Taking into account $\psi = \psi\ (s', C, \theta)$ and $ds'/dt = 0$, one has:

$$\rho\,\frac{de_{int}}{dt} = -\rho\,\theta\,\frac{\partial^2\psi}{\partial\theta^2}\frac{d\theta}{dt} + \rho\,Tr\,\left(\left(\frac{\partial\psi}{\partial C} - \theta\,\frac{\partial}{\partial\theta}\,\left(\frac{\partial\psi}{\partial C}\right)\right)\,\frac{dC}{dt}\right) \tag{11.47}$$

On the other hand, owing to (11.7) and (11.39), one has

$$\frac{\partial\zeta}{\partial C} = -\beta\,\frac{\partial\psi}{\partial C}$$

Hence, (11.30) becomes

$$\sigma_R = -2\rho\, F \,\frac{\partial \psi}{\partial C}\, F^T \ ,$$

and consequently:

$$\sigma_R - \theta\, \frac{\partial \sigma_R}{\partial \theta} = -2\rho\, F\, B\, F^T \ ,$$

where:

$$B = \frac{\partial \psi}{\partial C} - \theta\, \frac{\partial}{\partial \theta}\, \left(\frac{\partial \psi}{\partial C}\right) \ . \tag{11.48}$$

On the other hand, calculating as in the proof of Theorem 11.9, one has, for any 3×3 symmetric matrix B

$$\rho\ Tr\ \left(B\, \frac{dC}{dt}\right) = -\frac{2\rho}{\beta}\ Tr\ \left(F\, B\, F^T A\right)$$

where A is defined by

$$A = grad_s\ w - \frac{1}{2}\left(v\ \frac{\partial \beta}{\partial r} + grad\ \beta\ v^T\right) \ .$$

Applying this relation to B given by (11.48) leads to

$$\rho\ Tr\ \left(\ B\, \frac{dC}{dt}\right) = \frac{1}{\beta}\ Tr\ \left(\left(\sigma_R - \theta\ \frac{\partial \sigma_R}{\partial \theta}\right) A\right) \ . \tag{11.49}$$

Besides, one has:

$$\frac{\partial w}{\partial r} = \frac{\partial}{\partial r}\ (\beta v) = \beta\ \frac{\partial v}{\partial r} + v\ \frac{\partial \beta}{\partial r}$$

Hence, expression A becomes:

$$A = \beta\, grad_s v = \beta\, D \ . \tag{11.50}$$

Introducing (11.50) into (11.49) and this last expression into (11.47) gives

$$\rho\ \frac{de_{int}}{dt} = -\rho\, c_v\, \frac{d\theta}{dt} + Tr\ \left(\left(\sigma_R - \theta\ \frac{\partial \sigma_R}{\partial \theta}\right)\ D\right) \ .$$

where:

$$c_v = \theta\, \frac{\partial s}{\partial \theta} = -\theta\, \frac{\partial^2 \psi}{\partial \theta^2}$$

is the **heat capacity at constant volume**. Combining with the form (11.46) of the balance of energy leads to the **equation of heat transfer**:

$$\boxed{-\rho\, c_v\, \frac{d\theta}{dt} = \theta\ Tr\ \left(\frac{\partial \sigma_R}{\partial \theta}\ D\right) + Tr\ \left(\sigma_I\, D\right) - div\ h + \rho\ \frac{dq_I}{dt}\ .} \tag{11.51}$$

The physical interpretation of this equation is that the variation of reversible thermal energy, at the left hand member, is equal, at the right hand member, to the contributions of each term to the dissipation due to:

- the reversible stress variation resulting from the temperature one,
- the power of dissipation of the dissipative stresses,
- the heat flux,
- the irreversible heat sources.

We are now able to state the **second principle of the thermodynamics**:

Principle 11.11 *The* ***local production of entropy*** *of a continuous medium characterized by fields of temperature vector* $\hat{\mathbf{W}}$ *and momentum tensor* $\hat{\mathbf{T}}$ *is non negative:*

$$\Phi = \boldsymbol{Div}\ \left(\hat{\boldsymbol{T}}\ \hat{\vec{\boldsymbol{W}}}\right) - \left(\boldsymbol{e}^0(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right)\ \left(\boldsymbol{e}^0(\boldsymbol{T}_I(\vec{\boldsymbol{U}}))\right) \geq 0\ , \tag{11.52}$$

and vanishes if and only if the process is reversible.

In this expression, $\boldsymbol{e}^0$ is the time arrow and $\vec{\boldsymbol{U}}$ is the 4-velocity (see Section 9.2). As scalar field, the value of Φ is invariant. In terms of tensor fields, expression (11.53) is covariant, then consistent with Galileo's principle of relativity 1.13. Without gravitation and in any Galilean coordinate system, the expression of the local production of entropy is:

$$\Phi = div\ \left(\hat{T}\ \hat{W}\right) - \left(e^0\, f\, U\right)\ \left(e^0\, T_I U\right) \geq 0\ , \tag{11.53}$$

If the process is reversible, $T_I = 0$ and, because of Theorem 11.10:

$$\Phi = div\ \left(\hat{T}_R\ \hat{W}\right) = div\ S = \rho\,\frac{ds}{dt} = 0\ ,$$

thus the entropy is constant, that explains the name of Φ. Conversely, if the local production of entropy is positive, it cannot be proved that the process is dissipative, reason for which it is a principle –*i.e.* an axiom– and not a theorem.

Next, let us calculate explicitly the expression of the local production of entropy. Owing to (9.15), (11.8) and (1.12), the former factor of the second term of (11.53):

$$e^0\, f\, U = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial\beta}{\partial t} & \dfrac{\partial\beta}{\partial r} \\ \dfrac{\partial\, w}{\partial t} & \dfrac{\partial\, w}{\partial r} \end{pmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix} = \frac{\partial\beta}{\partial t} + \frac{\partial\beta}{\partial r}\, u = \frac{d\beta}{dt}\ ,$$

is invariant under any galilean transformation. Besides, (11.42) gives:

$$e^0\, T_I U = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{H}_I & 0 \\ h + \mathcal{H}_I v - \sigma_I v & \sigma_I \end{pmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix} = \mathcal{H}_I\ ,$$

which is a galilean invariant too. Thus the local production of entropy reads also:

$$\Phi = div\ \left(\hat{T}\ \hat{W}\right) - \mathcal{H}_I\,\frac{d\beta}{dt} \geq 0\ . \tag{11.54}$$

Now, we establish a new expression of the production of entropy.

Theorem 11.12 *If the momentum tensor $\hat{\mathbf{T}}$ is divergence free, the local production of entropy (11.53) is given by*

$$\Phi = h \ . \ grad \ \beta + \beta \ Tr \ (\sigma_I \ D) \geq 0 \ . \tag{11.55}$$

Proof Starting from (11.54), owing to (13.19) and the first principle 11.27, it holds:

$$\Phi = Tr \left(\hat{T} \ \frac{\partial \hat{W}}{\partial X} \right) - \mathcal{H}_I \frac{d\beta}{dt} = Tr \ (T \ f) + \frac{\partial \zeta}{\partial X} N + q_I \ \frac{d\beta}{dt} \ .$$

Because of theorem 11.9 $\diamondsuit$ or equivalently (11.32), one has:

$$\Phi = Tr \ (T \ f) - Tr \ (T_R \ f) + q_I \ \frac{d\beta}{dt} = Tr \ (T_I \ f) + q_I \ \frac{d\beta}{dt} \ .$$

Using expression (11.8) of the friction and (11.42) of the irreversible momentum tensor

$$Tr \ (T_I \ f) = -q_I \ \left(\frac{\partial \beta}{\partial t} + \frac{\partial \beta}{\partial r} \ v \right) + \frac{\partial \beta}{\partial r} \ h + Tr \ \left(\sigma_I \ \left(\frac{\partial w}{\partial r} \right) - v \ \frac{\partial \beta}{\partial r} \right) \ .$$

Owing to $w = \beta \ v$, it holds

$$Tr \ (T_I \ f) = -q_I \ \frac{d\beta}{dt} + h \ . \ grad \ \beta + \beta \ Tr \ \left(\sigma_I \ \frac{\partial v}{\partial r} \right) \ ,$$

and because σ_I is symmetric:

$$Tr \ (T_I \ f) + q_I \ \frac{d\beta}{dt} = h \ . \ grad \ \beta + \beta \ Tr \ (\sigma_I \ D) \ ,$$

that achieves the proof. ■

Through the relation:

$$\Phi = h \cdot a + Tr \ (\sigma_I A) \ ,$$

the interest of Theorem (11.12) is turning out a correspondance between:

- **thermodynamic forces** (or **affinities**) $a = grad \ \beta$, $A = \beta \ D$,
- and the corresponding **thermodynamic fluxes** h, σ_I.

$\hat{\mathbf{T}}_I$ being represented by $\tau_I = (h, \ \sigma_I)$ and $\mathbf{f}$ by $\alpha = (a, \ A)$, this pairing reads:

$$\Phi = \langle \tau_I, \ \varphi \rangle \ .$$

11.7 Constitutive laws in thermodynamics

To define completly the dissipative processes of the material, we need an additional relation called the constitutive law. In the most simple situations, it is given by a map $g : \alpha \mapsto \tau_I$, or more explicitly in terms of fluxes and affinities

$$(a, A) \mapsto (h, \sigma_I) = g\ (a, A)$$

Before discussing some aspects of the constitutive laws, we want to characterize the non dissipative processes thanks to the following proposition

Theorem 11.13 *For a continuum occupying a connected domain, let $g : \alpha \mapsto \tau_I$ be a continuous map defining a constitutive law and verifying the second principle*

$$\forall\ \alpha, \qquad \Phi = \langle g\ (\alpha),\ \alpha\rangle \geq 0\ , \tag{11.56}$$

Then, if the friction tensor field is identically null,

$\diamondsuit$ *the temperature field is uniform and the motion of the continuum is rigid,*

$\heartsuit$ *the heat conduction flux and the viscous stresses vanish.*

Proof As the friction is null, $a = grad\ \beta = 0$ then β and $\theta = 1/\beta$ are uniform on a connected domain. Besides, $A = \beta\, D = 0$ and $\beta > 0$, then $D = grad_s v = 0$. In a connected domain, there exist maps $t \mapsto v_0(t) \in \mathbb{R}^3$ and $t \mapsto \omega(t) \in \mathbb{R}^3$ and:

$$v\ (t, r) = v_0(t) + \omega\ (t) \times r\ ,$$

that defines a rigid motion of the continuum and proves $\diamondsuit$.

If λ is a real number, the condition (11.56) gives:

$$\langle g\ (\lambda\alpha),\ \lambda\alpha\rangle = \lambda\ \langle g\ (\lambda\alpha),\ \alpha\rangle \geq 0\ ,$$

which means that λ and $\langle g\ (\lambda\alpha),\ \alpha\rangle$ have the same sign. As λ tends to 0, by continuity:

$$\langle g\ (0),\ \alpha\rangle = 0\ .$$

As this occurs for any α, it is possible only if $\tau_I = g\ (0) = 0$. Then h and σ_I vanish, that proves $\heartsuit$. ■

Our aim is now to find the explicit form of the constitutive law in relatively simple situations, for instance when the behaviour of the continuum is isotropic and the law is linear. First of all, we have to discuss how the components of $\mathbf{f}$ and $\mathbf{T_I}$ change under galilean and bargmannian transformations. Let us consider a galilean transformation with boost v and rotation R:

$$P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}$$

The tensorial rule of 1-covariant and 1-contravariant tensors gives for f:

$$f' = P^{-1} f \, P \ ,$$

then:

$$\frac{\partial \beta'}{\partial t'} = \frac{\partial \beta}{\partial t} + \frac{\partial \beta}{\partial r} u \ ,$$

$$\frac{\partial \beta'}{\partial r'} = \frac{\partial \beta}{\partial r} R \ , \tag{11.57}$$

$$\frac{\partial w'}{\partial t'} = R^T \left(\frac{\partial w}{\partial t} + \frac{\partial w}{\partial r} u \right) - \left(\frac{\partial \beta}{\partial t} + \frac{\partial \beta}{\partial r} u \right) R^T u \ ,$$

$$\frac{\partial w'}{\partial r'} = R^T \left(\frac{\partial w}{\partial r} - u \, \frac{\partial \beta}{\partial r} \right) \, R \ . \tag{11.58}$$

By transposing relation (11.57), one has:

$$a' = R \, a \ . \tag{11.59}$$

Also, taking into account (11.57), (11.58) and the velocity addition formula(1.13), one gets:

$$\frac{\partial w'}{\partial r'} - v' \, \frac{\partial \beta'}{\partial r'} = R^T \left(\frac{\partial w}{\partial r} - v \, \frac{\partial \beta}{\partial r} \right) \, R \ .$$

Hence, the transformation law of (11.50) is

$$A' = R^T A \, R \ . \tag{11.60}$$

Now, we can determine the invariants of α. It is easy to verifiy that there are the 3 eigenvalues of A, $\parallel a \parallel$, $\parallel A \ a \parallel$ and $a^T A \ a$.

The transformation laws (11.44) and (11.19) of σ_I and h are formally the same as the ones (11.59) and (11.60) of a and A. By analogy with α, the 6 independent invariants of τ_I are the 3 eigenvalues of σ_I, $\parallel h \parallel$, $\parallel \sigma_I h \parallel$ and $h^T \sigma_I \ h$. Once again, we can verify that the production of entropy is invariant:

$$h' \cdot a' + Tr \ (\sigma'_I A') = h \cdot a + Tr \ (\sigma_I A) \ .$$

On this ground, we can construct constitutive laws. For instance, an isotropic linear law has the form:

$$h = k_1 a \ , \tag{11.61}$$

$$\sigma_I = k_2 \ Tr \ (A) \ I_{\mathbb{R}^3} + k_3 \ A \ , \tag{11.62}$$

where $k_1, k_2, k_3 \in \mathbb{R}$. Introducing (11.61) and (11.62) into the production of entropy (11.55) gives:

$$\Phi = k_1 \ \parallel a \parallel^2 + k_2 \ (Tr \ A)^2 + k_3 \ Tr \ \left(A^2 \right) \ ,$$

which is satisfied if the following restrictions are imposed to the material parameters:

$$k_1 \geq 0, \qquad k_3 \geq 0, \qquad k_2 + \frac{k_3}{3} \geq 0 \ .$$

Law 11.14 *In terms of temperature, the constitutive laws are:*

- ***Fourier's law*** *or* ***law of heat conduction****:*

$$\boxed{h = -k \, grad \ \theta \ ,} \tag{11.63}$$

where $k = k_1 \ / \ \theta^2 \geq 0$ is the ***thermal conductivity****,*

- ***Newton's viscous flow law****:*

$$\boxed{\sigma_I = \eta \, (div \ v) \ I_{\mathbb{R}^3} + 2\mu \, grad_s \ v \ ,} \tag{11.64}$$

where $\eta = k_2 \ / \ \theta$ and $\mu = k_3 \ / \ 2\,\theta \geq 0$ is the ***dynamic viscosity****.*

For simple fluids (in particular: water, air, gases as methane), the law can be simplified by assuming that the viscous stresses σ_I are traceless (**Stokes hypothesis**), leading to:

$$\sigma_I = 2\mu \left(grad_s \ v - \frac{1}{3} \, (div \ v) \ I_{\mathbb{R}^3} \right) \ .$$

In many situations, the material parameters k and η are considered to be constant. For barotropic fluids, owing to (9.65) and (13.13), one has:

$$(div \, \sigma_R)^T = -grad \, q \ .$$

On the other hand, Newton's viscous flow law (11.64) combined with (13.20), (13.15) and (13.13) gives:

$$(div \, \sigma_I)^T = \mu \, \triangle v + \frac{\mu}{3} \, grad \, (div \, v) \ .$$

Introducing the two previous expressions into the balance of linear momentum (Theorem 11.7, ♡) with $\sigma = \sigma_R + \sigma_I$, we obtain **Navier-Stokes equations**:

$$\boxed{\rho \ \left[\frac{\partial v}{\partial t} + \frac{\partial v}{\partial r} \, v \right] = -grad \, q + \mu \, \triangle v + \frac{\mu}{3} \, grad \, (div \, v) \ .} \tag{11.65}$$

Definition 11.15 *A fluid is* ***incompressible*** *if the mass of each volume element remains constant :*

$$\frac{d\rho}{dt} = 0 \ ,$$

then the density ρ is an integral of the motion and, owing to (9.53):

$$div \, v = 0 \ .$$

Also, taking into account (13.24) and (9.75), J is an integral of the motion:

$$\frac{dJ}{dt} = J\,Tr\left(\frac{dF}{dt}\,F^{-1}\right) = J\,div\,v = 0 \ .$$

Hence, in (9.74), the derivative has no sense and the pressure q is an undeterminate. The **Navier-Stokes equations for incompressible flows** are:

$$\boxed{div\,v = 0, \qquad \rho\,\left[\frac{\partial v}{\partial t} + \frac{\partial v}{\partial r}\,v\right] = -grad\,q + \mu\,\triangle v \ .} \tag{11.66}$$

11.8 Thermodynamics and Galilean gravitation

Until now, we are concerned only with the uniform straight motion which can be described by the calculus of variation with a Lagrangian equal to the kinetic energy as mentioned at Section 6.2. It was also seen that this expression of the Lagrangian is not general and, for a particle subjected to a Galilean gravitation, it must be replaced by (6.12):

$$\mathcal{L}(t, r, v) = \frac{1}{2}\,m\parallel v \parallel^2 - m\,\phi + m\,A\cdot v \ ,$$

containing the gravitation potential ϕ and A. Introducing a coordinate system $\hat{X}'$ for which one:

$$dz' = \frac{\mathcal{L}}{m}\,dt \ .$$

This extra coordinate z' has the physical dimension and the meaning of an **action by unit mass**. Taking into account (11.3), we obtain:

$$dz' = dz - \phi\,dt + A\cdot dr \ ,$$

which can be completed by:

$$dt' = dt, \qquad dr' = dr \ ,$$

to define a linear transformation:

$$d\hat{X}' = \hat{Q}^{-1}\,d\hat{X} \ .$$

where:

$$\hat{Q}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_{\mathbb{R}^3} & 0 \\ -\phi & A^T & 1 \end{pmatrix}, \qquad \hat{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_{\mathbb{R}^3} & 0 \\ \phi & -A^T & 1 \end{pmatrix} \ . \tag{11.67}$$

What is the physical interpretation of these transformations? Before applying it, the particle is, in absence of gravitation, in Uniform Straight Motion (USM). The effect of applying such a transformation is to embed the particle in the gravitation field. It is straightforward to verify that the set of such transformation matrix is an abelian subgroup of the affine group $\mathbb{A}ff(5)$. It is also worth to notice that, according to the transformation law (13.2) of 2-covariant tensors, Gram's matrix of the covariant metric tensor $\hat{\mathbf{G}}$ in the new coordinate system $\hat{X}'$ is given by:

$$\hat{G}' = Q^T \hat{G}\, Q \ ,$$

that gives for the metric embedded in the gravitation field:

$$\hat{G}' = \begin{pmatrix} -2\phi & A^T & -1 \\ A & 1_{\mathbb{R}^3} & 0 \\ -1 & 0 & 0 \end{pmatrix} .$$

The attentive reader will observe that reducing this matrix to the space-time provides the one given by (9.17). Our starting point is now to work in these coordinate systems $\hat{X}'$ that we call **Bargmannian coordinate systems**. When equipped with the previous covariant metric, the 5-dimensional space $\hat{\mathcal{U}}$ is now a riemannian manifold and $\mathcal{U}$ a 4-dimensional submanifold thereof.

There exists one and only one symmetric covariant differential such that the covariant differential of the metric vanishes. The only non vanishing quantities (13.35) are (10.4), (10.5). Using (13.36) and taking into account the definition (6.13) of galilean gravitation potentials, the only non vanishing Christoffel's symbols are:

$$\Gamma^j_{00} = -g^j, \qquad \Gamma^j_{0k} = \Gamma^j_{k0} = \Omega^j_k \ , \tag{11.68}$$

$$\Gamma^4_{00} = \frac{\partial \phi}{\partial t} - A \cdot g, \qquad \Gamma^4_{ij} = \frac{1}{2}\left(\frac{\partial A_i}{\partial r^j} + \frac{\partial A_j}{\partial r^i}\right) = (grad_s A)^i_j \ , \tag{11.69}$$

$$\Gamma^4_{0i} = \Gamma^4_{i0} = \frac{\partial \phi}{\partial r^i} - \frac{1}{2}\left(\frac{\partial A_i}{\partial r^j} - \frac{\partial A_j}{\partial r^i}\right) A^j = (grad\, \phi - \Omega \times A)^i \ . \tag{11.70}$$

It is worth to observe that we recover (9.47). In matrix form, the gravitation of the space $\hat{\mathcal{U}}$ reads:

$$\hat{\Gamma}(d\hat{X}) = \begin{pmatrix} 0 & 0 & 0 \\ j(\Omega)\, dr - g\, dt & j(\Omega)\, dt & 0 \\ \left(\frac{\partial \phi}{\partial t} - A \cdot g\right) dt + (grad\, \phi - \Omega \times A) \cdot dr & [(grad\, \phi - \Omega \times A)\ dt - grad_s A\, dr]^T & 0 \end{pmatrix},$$

It is the expansion of the space-time gravitation (3.38) to the fifth dimension.

In a similar way, the thermodynamical tensors can be embedded into the gravitation field. Applying the matrix $\hat{Q}^{-1}$ given by (11.67) preserves β, w and embeds the ζ component in the gravitation, that reads omitting the bar:

$$\zeta = \zeta_{int} + \frac{1}{2\beta} \parallel w \parallel^2 - \beta\, \phi + A \cdot w \ .$$

Taking into account(13.29), it is represented in a Galilean coordinate system by the 4×4 matrix

$$f = \nabla W = \begin{pmatrix} \dfrac{\partial \beta}{\partial t} & \dfrac{\partial \beta}{\partial r} \\ \\ \dfrac{\partial\, w}{\partial t} - \beta g + \Omega \times w & \dfrac{\partial\, w}{\partial r} + \beta j(\Omega) \end{pmatrix} . \qquad (11.71)$$

Let us calculate the expression of the momentum tensor enbedded in the gravitation field. The transformation law (11.10) reads:

$$\hat{T}' = Q^{-1} \hat{T}\, \hat{Q} \ ,$$

with $\hat{T}$ is given by (11.20), $\hat{Q}$ is given by (11.67) and the corresponding Q is $1_{\mathbb{R}^4}$, that leads to:

$$\hat{T}' = \begin{pmatrix} \mathcal{H}' & -\pi^T & \rho \\ h + \mathcal{H}' v - \sigma v & \sigma - v \pi^T & \rho v \end{pmatrix} , \qquad (11.72)$$

where occurs :

- the hamiltonian density: $\mathcal{H}' = \rho\, \left(\frac{1}{2} \parallel v \parallel^2 + \phi + e_{int} - q_I \right)$,
- the generalized linear momentum: $\pi = \rho\, (v + A)$.

The attentive reader will observe that we recover by a new way the expression (9.79) or (10.24) of the hamiltonian density and the expression (10.23) of π. In the sequel, we are implicitly supposed to work in Bargmannian coordinate systems but the prime shall be omitted for sake of easiness. Let us examine now the expression of the first principle of the thermodynamics (11.27) in presence of gravitation.

Theorem 11.16 *If $\hat{T}$ is covariant divergence free:*

$$Div_X\, \hat{T} = 0 \ ,$$

then, we have:

♢ ***balance of mass:*** $\dfrac{\partial \rho}{\partial t} + div\, (\rho\, v) = 0$,

♡ ***balance of linear momentum:*** $\rho\, \left[\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial r}\, v \right] = (div\ \sigma)^T + \rho\, \left(g - 2\, \Omega \times v) \right)$,

♠ ***balance of energy:*** $\dfrac{\partial \mathcal{H}}{\partial t} + div\ \left(h + \mathcal{H} v - \sigma v \right) = \rho \left(\dfrac{\partial \phi}{\partial t} - \dfrac{\partial A}{\partial t} \cdot v \right)$.

Proof. As the space-time $\mathcal{U}$ is a submanifold of the fifth dimensional space $\hat{\mathcal{U}}$, an event $\boldsymbol{X} \in \mathcal{U}$ belongs also to $\hat{\mathcal{U}}$. We consider a momentum field $\boldsymbol{X} \mapsto \hat{\boldsymbol{T}}(\boldsymbol{X})$ where

$\hat{\boldsymbol{T}}(\boldsymbol{X})$ is a 1-covariant tensor on the tangent space $T_{\boldsymbol{X}}\hat{\mathcal{U}}$ with vector values in $T_{\boldsymbol{X}}\mathcal{U}$ (which can be identified to a linear map from $T_{\boldsymbol{X}}\hat{\mathcal{U}}$ to $T_{\boldsymbol{X}}\mathcal{U}$). We wish to calculate its covariant divergence. In addition to Convention 1.1, we adopt the extra one: Greek indices $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and so on run over the five coordinate labels $0, 1, 2, 3, 4$. The basis $(\vec{\boldsymbol{e}}_\alpha)$ of $T_{\boldsymbol{X}}\mathcal{U}$ is completed by $\vec{\boldsymbol{e}}_4$ to build a basis $(\vec{\boldsymbol{e}}_{\hat{\alpha}})$ of $T_{\boldsymbol{X}}\hat{\mathcal{U}}$ in which the momentum tensor is decomposed as:

$$\hat{\boldsymbol{T}} = \hat{T}^\gamma \vec{\boldsymbol{e}}_\gamma, \qquad \hat{T}^\gamma = \hat{T}^\gamma_{\hat{\alpha}} \boldsymbol{e}^{\hat{\alpha}} \ .$$

Taking into account (13.28), its covariant differential is:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} \hat{\boldsymbol{T}} = \nabla_{\overrightarrow{d\boldsymbol{X}}} (\hat{T}^\gamma \vec{\boldsymbol{e}}_\gamma) = (\nabla_{\overrightarrow{d\boldsymbol{X}}} \hat{T}^\gamma)\, \vec{\boldsymbol{e}}_\gamma + \hat{T}^\gamma\, (\nabla_{\overrightarrow{d\boldsymbol{X}}} \vec{\boldsymbol{e}}_\gamma) = (\nabla_{\overrightarrow{d\boldsymbol{X}}} \hat{T}^\gamma + \Gamma^\gamma_\rho \hat{T}^\rho)\, \vec{\boldsymbol{e}}_\gamma \ ,$$

where, owing to (13.34):

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} \hat{T}^\gamma = \nabla_{dX} (\hat{T}^\gamma_{\hat{\alpha}} \boldsymbol{e}^{\hat{\alpha}}) = d\hat{T}^\gamma_{\hat{\alpha}}\, \boldsymbol{e}^{\hat{\alpha}} + \hat{T}^\gamma_{\hat{\alpha}}\, \nabla_{dX}\, \boldsymbol{e}^{\hat{\alpha}} = (d\hat{T}^\gamma_{\hat{\alpha}} - \hat{T}^\gamma_{\hat{\beta}} \Gamma^{\hat{\beta}}_{\hat{\alpha}})\, \boldsymbol{e}^{\hat{\alpha}} \ .$$

Hence, we obtain:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} \hat{\boldsymbol{T}} = \left[\nabla_{dX} \hat{T}^\gamma_{\hat{\alpha}}\, \boldsymbol{e}^{\hat{\alpha}} \right] \vec{\boldsymbol{e}}_\gamma \ ,$$

with:

$$\nabla_{dX} \hat{T}^\gamma_{\hat{\alpha}} = d\hat{T}^\gamma_{\hat{\alpha}} + \Gamma^\gamma_\rho \hat{T}^\rho_{\hat{\alpha}} - \hat{T}^\gamma_{\hat{\beta}} \Gamma^{\hat{\beta}}_{\hat{\alpha}} \ ;$$

Hence, there exists a field $\nabla \hat{\boldsymbol{T}}$ of 2-covariant and 1-contravariant tensors such that:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} \hat{\boldsymbol{T}} = (\nabla \hat{\boldsymbol{T}}) \cdot \overrightarrow{d\boldsymbol{X}} \ .$$

Using Christoffel's symbols (13.30), one has:

$$\nabla \hat{\boldsymbol{T}} = \left[\nabla_\sigma \hat{T}^\gamma_{\hat{\alpha}}\, \boldsymbol{e}^{\hat{\alpha}} \right] \vec{\boldsymbol{e}}_\gamma \otimes \boldsymbol{e}^\sigma \ ,$$

with:

$$\nabla_\sigma \hat{T}^\gamma_{\hat{\alpha}} = \frac{\partial \hat{T}^\gamma_{\hat{\alpha}}}{\partial X^\sigma} + \Gamma^\gamma_{\rho\sigma} T^\rho_{\hat{\alpha}} - \hat{T}^\gamma_{\hat{\beta}} \Gamma^{\hat{\beta}}_{\hat{\alpha}\sigma} \ .$$

By contraction, we define the **covariant divergence** of the momentum tensor:

$$\boldsymbol{Div}\, \hat{\boldsymbol{T}} = \nabla_\gamma \hat{T}^\gamma_{\hat{\alpha}}\, \boldsymbol{e}^{\hat{\alpha}} \ ,$$

with:

$$\nabla_\gamma \hat{T}^\gamma_{\hat{\alpha}} = \frac{\partial \hat{T}^\gamma_{\hat{\alpha}}}{\partial X^\gamma} + \Gamma^\gamma_{\rho\gamma} T^\rho_{\hat{\alpha}} - \hat{T}^\gamma_{\hat{\beta}} \Gamma^{\hat{\beta}}_{\hat{\alpha}\gamma} \ . \tag{11.73}$$

In indicial notation, the components of $\hat{\boldsymbol{T}}$ are:

$$\hat{T}^0_0 = \mathcal{H}, \qquad \hat{T}^0_i = -\delta_{ik}\pi^k, \qquad \hat{T}^0_4 = \rho \ , \tag{11.74}$$

$$\hat{T}^j_0 = h^j + \mathcal{H} v^j - \sigma^j_k v^k, \qquad \hat{T}^j_i = \sigma^j_i - v^j \delta_{ik} \pi^k, \qquad \hat{T}^j_4 = p^j \ . \tag{11.75}$$

The first principle of the thermodynamics (11.27) reads:

$$\nabla_\gamma \hat{T}^\gamma_{\hat{\alpha}} = 0 \ ,$$

where Christoffel's symbols are given by (11.68), (11.69) and (11.70). Putting $\hat{\alpha} = 4$ in the previous equation and taking into account the vanishing terms, one has:

$$\nabla_\gamma \hat{T}^\gamma_4 = \frac{\partial \hat{T}^\gamma_4}{\partial X^\gamma} = 0 \ ,$$

that allows to recover the balance of mass $\diamondsuit$. Similarly, putting $\hat{\alpha} = i$ and taking into account the non vanishing terms, it holds:

$$\nabla_\gamma \hat{T}^\gamma_i = \frac{\partial \hat{T}^\gamma_i}{\partial X^\gamma} - \hat{T}^0_j \Gamma^j_{i0} - \hat{T}^0_4 \Gamma^4_{i0} - \hat{T}^j_4 \Gamma^4_{ij} = 0 \ ,$$

or, owing to the momentum components (11.74) and (11.75):

$$-\frac{\partial}{\partial t}\left(\delta_{ik}\pi^k\right) + \frac{\partial}{\partial r^j}\left(\sigma^j_i - v^j \delta_{ik}\pi^k\right) + \delta_{jk}\pi^k \Omega^j_i - \rho\, \Gamma^4_{i0} - p^j \Gamma^4_{ij} = 0 \ .$$

Owing to (13.14), it reads in matrix form:

$$-\frac{\partial}{\partial t}(\rho\,(v+A)^T) + div\,\sigma - div\,(\rho v)\,(v+A)^T - \rho v^T grad\,(v+A)$$

$$+\rho\,(v+A)^T j(\Omega) - \rho\,\left(grad\,\phi - \Omega \times A\right)^T + \rho\, v^T grad_s\, A = 0 \ .$$

But, owing to (6.13), it holds:

$$grad_s\, A = \frac{\partial A}{\partial r} - j(\Omega) \ ,$$

then, taking into account (10.26), we recover (10.27) and finishing the calculation as in the proof of Theorem 10.1, we demonstrate the balance of linear momentum $\heartsuit$.

Finally, putting $\hat{\alpha} = 0$ and taking into account the non vanishing terms, it holds:

$$\nabla_\gamma \hat{T}^\gamma_0 = \frac{\partial \hat{T}^\gamma_0}{\partial X^\gamma} - \hat{T}^0_j \Gamma^j_{00} - \hat{T}^i_j \Gamma^4_{i0} - \hat{T}^0_4 \Gamma^4_{00} - \hat{T}^j_4 \Gamma^4_{j0} = 0 \ ,$$

which reads in matrix form, after some simplifications:

$$\frac{\partial \mathcal{H}}{\partial t} + div\ \left(h + \mathcal{H} v - \sigma v\right) - \rho \left(v \cdot (g + grad\,\phi) + \frac{\partial \phi}{\partial t} \right) = 0 \ .$$

But, owing to (6.13), it holds:

$$v \cdot g + \frac{d\phi}{dt} = v \cdot (g + grad\,\phi) + \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial t} - v \cdot \frac{\partial A}{\partial t} \qquad (11.76)$$

which leads to the balance of energy ♠ and achieves the proof. ∎

Theorem 11.9 concerning the reversible processes remains true. Only the demonstration of ♢ is different. Before calculating, it is worth noting that the covariant derivative of ζ is meaningfull because ζ is not a scalar but a component of the temperature vector. We can verify that:

$$(\nabla\zeta)\, N = \rho \left(\frac{d\zeta}{dt} - \beta g \cdot v \right) \ .$$

the remaing part of the calculation is straightforward taking into account the expression (11.71) of the friction embedded into the gravitation field.

For the dissipative continua, the equation of heat transfer (11.51) is slightly modified. Wet let the reader to show that (11.46) is replaced by:

$$\rho\, \frac{d}{dt}\,(e_{int} + \phi) = Tr\ \ (\sigma\, D) - div\ h + \rho\, \frac{dq_I}{dt} + \rho \left(\frac{\partial \phi}{\partial t} - \frac{\partial A}{\partial t} \cdot v \right) \ ,$$

or, taking into account (11.76):

$$\rho\, \frac{de_{int}}{dt} = Tr\ \ (\sigma\, D) + g \cdot v - div\ h + \rho\, \frac{dq_I}{dt} \ .$$

Next, the **equation of heat transfer with gravitation** is:

$$\boxed{-\rho\, c_v\, \frac{d\theta}{dt} = \theta\ Tr\ \left(\frac{\partial \sigma_R}{\partial \theta}\, D \right) + Tr\ \ (\sigma_I\, D) + g \cdot v - div\ h + \rho\, \frac{dq_I}{dt} \ .} \qquad (11.77)$$

Taking into account the expression (11.71) of the friction, the reader can also easily verify that the expression (11.54) of the production of entropy is replaced by:

$$\Phi = Div\ \left(\hat{T}\ \hat{W} \right) - \mathcal{H}_I\, \frac{d\beta}{dt} \geq 0 \ , \qquad (11.78)$$

where occurs now the covariant divergence in the first term. To be consistent with Galileo's principle of relativity (1.13), Theorem 11.12 is replaced by:

Theorem 11.17 *If the momentum tensor* $\hat{\mathbf{T}}$ *is* ***covariant*** *divergence free, the local production of entropy (11.53) is given by 11.55*

$$\Phi = h\ .\ grad\ \beta + \beta\ Tr\ \ (\sigma_I\ D) \geq 0 \ .$$

Proof According to the rule (13.33), one has:

$$\boldsymbol{Div}\,(\hat{\boldsymbol{T}}\cdot\hat{\boldsymbol{W}}) = (\boldsymbol{Div}\,\hat{\boldsymbol{T}})\cdot\hat{\boldsymbol{W}} + \hat{\boldsymbol{T}} : \nabla\hat{\boldsymbol{W}} \ ,$$

or in local coordinates:

$$Div(\hat{T}\,\hat{W}) = (Div\,\hat{T})\,\hat{W} + Tr\,(\hat{T}\,\nabla\hat{W}) \ ,$$

hence, starting from (11.78) and owing to the first principle (11.27), it holds:

$$\Phi = Tr\ \left(\hat{T}\,\nabla\hat{W}\right) - \mathcal{H}_I\frac{d\beta}{dt} = Tr\ \left(T\ f\right) + (\nabla\zeta)\,N + q_I\ \frac{d\beta}{dt}\ .$$

Because of theorem 11.9 ◊ or equivalently (11.31), one has:

$$\Phi = Tr\ \left(T_I\ f\right) + q_I\ \frac{d\beta}{dt}\ ,$$

and the proof follows the one of Theorem 11.12, that achieves the proof. ■

11.9 Exercises

11.9.1 Local production of entropy

Starting from (11.54) and using the balance of mass, show that [Comment 1]:

$$\Phi = \rho\ \frac{ds}{dt} - \frac{\rho}{\theta}\ \frac{dq_I}{dt} + div\ \left(\frac{h}{\theta}\right) \geq 0\ . \tag{11.79}$$

11.10 Comments for experts

[Comment 1] This relation is known in the literature as Clausius-Duhem inequality but it seems first appearing in Truesdell's works ([38], [39])

Part III

From Elementary Algebra to Differential Geometry

Chapter 12

Elementary mathematical tools

12.1 Maps

We call **map** an assignment $f : x \mapsto y = f(x)$ of elements of a set into elements of another set. The existence of a map f entails the one of its **definition set**:

$$x \in def(f) \qquad \Leftrightarrow \qquad f(x) \quad \text{exists} \ ,$$

and of its **value set**:

$$y \in val(f) \qquad \Leftrightarrow \qquad \exists x \quad \text{such that} \quad y = f(x) \ .$$

It is useful to consider the **identity** of a set E, denoted 1_E and the **impotent map** of which the definition set is empty, that we denote $1_\emptyset$. The **composition** of functions is defined by:

$$(f \circ g)(x) = f(g(x)) \ ,$$

every time it exists. If there is no confusion, the composition $f \circ g$ is simply denoted fg. The composition of two maps is always a map, even if it is impotent. The composition is associative but not commutative. A map f is **regular** or **one-to-one** if:

$$f(x) = f(y) \qquad \Leftrightarrow \qquad x = y \ .$$

f being regular, the **inverse** map f^{-1} is defined by:

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad y = f(x) \ .$$

We verify immediately that f^{-1} is regular and:

$$(f^{-1})^{-1} = f, \quad def(f^{-1}) = val(f), \quad val(f^{-1}) = def(f) \ ,$$

$$f^{-1}f = 1_{def(f)}, \quad ff^{-1} = 1_{val(f)} \ .$$

If f and g are regular, fg is regular and:

$$(fg)^{-1} = g^{-1}f^{-1} \ .$$

Two maps f and g being defined on the same set, there exists a map h such that $f = hg$ if and only if:

$$\{g(x) = g(y)\} \qquad \Rightarrow \qquad \{f(x) = f(y)\} \ .$$

Besides, the map h is unique if $def(h) = val(g)$. Hence it is denoted f/g and is called the **quotient** of f by g.

A set $\mathcal{R}$ is a **pre-collection** if:

$$f, g \in \mathcal{R} \qquad \Rightarrow \qquad f^{-1}, fg \in \mathcal{R} \ .$$

We call **space** of $\mathcal{R}$ the union of the definition sets of the elements of $\mathcal{R}$.

12.2 Matrix calculus

12.2.1 Columns

A n-**column** or simply a **column** is an element of $\mathbb{R}^n$ and is denoted:

$$V = \begin{pmatrix} V^1 \\ V^2 \\ \vdots \\ V^n \end{pmatrix} \ .$$

V^i are its **components**. $\mathbb{R}^n$ has two operations, the addition of vectors and the multiplication by a scalar:

$$\begin{pmatrix} U^1 \\ U^2 \\ \vdots \\ U^n \end{pmatrix} + \begin{pmatrix} V^1 \\ V^2 \\ \vdots \\ V^n \end{pmatrix} = \begin{pmatrix} U^1 + V^1 \\ U^2 + V^2 \\ \vdots \\ U^n + V^n \end{pmatrix} \qquad \text{and} \qquad \lambda \begin{pmatrix} V^1 \\ V^2 \\ \vdots \\ V^n \end{pmatrix} = \begin{pmatrix} \lambda V^1 \\ \lambda V^2 \\ \vdots \\ \lambda V^n \end{pmatrix} \ .$$

The addition is associative and commutative. The multiplication by a scalar is distributive over the column addition. We call **key-columns** the following elements of $\mathbb{R}^n$:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \ .$$

Thus, any n-column is a linear combination of the key-columns:

$$V = V^1 e_1 + V^1 e_2 + \cdots + V^n e_n \ .$$

The n-columns $V_1, V_2, \cdots, V_p$ are **linearly independent** if any linear combination $\lambda_1 V_1 + \lambda_2 V_2 + \cdots + \lambda_p V_p$ is zero if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_p = 0$.

12.2.2 Rows

A n-**row** or simply a **row** is a numerical linear function on $\mathbb{R}^n$:

$$\Phi(\lambda U + \mu V) = \lambda\Phi(U) + \mu\Phi(V) \ .$$

It is denoted:

$$\Phi = \begin{pmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_n \end{pmatrix} \ .$$

Φ_i are its **components** and the value of Φ for the n-column V is:

$$\Phi(V) = \Phi_1 V^1 + \Phi_1 V^1 + \cdots + \Phi_n V^n \ .$$

The set $(\mathbb{R}^n)^*$ of the rows has two operations, the addition:

$$\begin{pmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_n \end{pmatrix} + \begin{pmatrix} \Theta_1 & \Theta_2 & \cdots & \Theta_n \end{pmatrix} = \begin{pmatrix} \Phi_1 + \Theta_1 & \Phi_2 + \Theta_2 & \cdots & \Phi_n + \Theta n \end{pmatrix} ,$$

and the multiplication by a scalar:

$$\lambda \begin{pmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_n \end{pmatrix} = \begin{pmatrix} \lambda\Phi_1 & \lambda\Phi_2 & \cdots & \lambda\Phi_n \end{pmatrix} \ .$$

The addition is associative and commutative. The multiplication by a scalar is distributive over the row addition. We call **key-rows** the following rows:

$$e^1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} , \quad e^2 = \begin{pmatrix} 0 & 1 & \cdots & 0 \end{pmatrix} , \quad \ldots , \quad e^n = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} \ .$$

Thus, any n-row is a linear combination of the key-rows:

$$\Phi = \Phi_1 e^1 + \Phi_2 e^2 + \cdots + \Phi_n e^n \ .$$

12.2.3 Matrices

A $n \times p$ **matrix** is a linear map from $\mathbb{R}^n$ into $\mathbb{R}^p$:

$$M(\lambda U + \mu V) = \lambda M(U) + M\Phi(V) \ .$$

It is denoted:

$$M = \begin{pmatrix} M_1^1 & M_2^1 & \cdots & M_p^1 \\ M_1^2 & M_2^2 & \cdots & M_p^2 \\ \vdots & \vdots & \ddots & \vdots \\ M_1^n & M_2^n & \cdots & M_p^n \end{pmatrix} \ .$$

M_j^i are its **elements**. The set $\mathbb{M}_{np}$ of the $n \times p$ matrices has two basic operations, the addition:

$$\begin{pmatrix} M_1^1 & \cdots & M_p^1 \\ \vdots & \ddots & \vdots \\ M_1^n & \cdots & M_p^n \end{pmatrix} + \begin{pmatrix} N_1^1 & \cdots & N_p^1 \\ \vdots & \ddots & \vdots \\ N_1^n & \cdots & N_p^n \end{pmatrix} = \begin{pmatrix} M_1^1 + N_1^1 & \cdots & M_p^1 + N_p^1 \\ \vdots & \ddots & \vdots \\ M_1^n + N_1^n & \cdots & M_p^n + N_p^n \end{pmatrix} ,$$

and the multiplication by a scalar:

$$\lambda \begin{pmatrix} M_1^1 & \cdots & M_p^1 \\ \vdots & \ddots & \vdots \\ M_1^n & \cdots & M_p^n \end{pmatrix} = \begin{pmatrix} \lambda M_1^1 & \cdots & \lambda M_p^1 \\ \vdots & \ddots & \vdots \\ \lambda M_1^n & \cdots & \lambda M_p^n \end{pmatrix} .$$

The matrix addition is associative and commutative. The multiplication by a scalar is distributive over the matrix addition. The j-th column of M is the column M_j of which the components are M_j^i, that allows to write in a more compact way:

$$M = \begin{pmatrix} M_1 & \cdots & M_n \end{pmatrix} .$$

The i-th row of M is the row M^i of which the components are M_j^i, that allows to write:

$$M = \begin{pmatrix} M^1 \\ \vdots \\ M^n \end{pmatrix} .$$

The composition or product of the $n \times p$ matrix M by the $p \times q$ matrix N is the $n \times q$ matrix obtained performing products "rows by columns":

$$MN = \begin{pmatrix} M^1 N_1 & M^1 N_2 & \cdots & M^1 N_p \\ M^2 N_1 & M^2 N_2 & \cdots & M^2 N_p \\ \vdots & \vdots & \ddots & \vdots \\ M^n N_1 & M^n N_2 & \cdots & M^n N_p \end{pmatrix} .$$

The matrix product is associative but not commutative. The **identity matrix**:

$$1_{\mathbb{R}^n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} ,$$

is such that for any $n \times n$ matrix M:

$$M \, 1_{\mathbb{R}^n} = 1_{\mathbb{R}^n} M = M$$

The components of the identity matrix are denoted δ_j^i and called **Kronecker's symbols**. A $n \times n$ matrix M is **diagonal** if it has the form:

$$M = \begin{pmatrix} M_1^1 & 0 & \cdots & 0 \\ 0 & M_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_n^n \end{pmatrix} .$$

We denote it:

$$M = diag(M_1^1, M_2^2, \cdots, M_n^n) .$$

The **transposed matrix** of the $n \times p$ matrix M is the $p \times n$ matrix:

$$M^T = \begin{pmatrix} M_1^1 & M_1^2 & \cdots & M_1^n \\ M_2^1 & M_2^2 & \cdots & M_2^n \\ \vdots & \vdots & \ddots & \vdots \\ M_p^1 & M_p^2 & \cdots & M_p^n \end{pmatrix} .$$

We verify that:

$$(M^T)^T = M \ .$$

A $n \times n$ matrix M is **symmetric** if $M^T = M$ and **skew-symmetric** if $M^T = -M$.

Of course, as particular cases:

- n-columns are $n \times 1$ matrices (then linear maps from $\mathbb{R}$ into $\mathbb{R}^n$) and $\mathbb{R}^n = \mathbb{M}_{n1}$,
- n-rows are $1 \times n$ matrices and $(\mathbb{R}^n)^* = \mathbb{M}_{1n}$. Moreover: $\Phi(V) = \Phi V$,
- scalars are 1×1 matrices and commute with any other matrices, in particular with columns and rows.

The **dot product** of two n-column is the scalar:

$$U \cdot V = U^T V \ .$$

The dot product is commutative. The **norm** of a n-column is:

$$\| U \| = \sqrt{U \cdot U} \ .$$

The norm is non negative. It vanishes if and only if the column vanishes. For any n-columns U, V and scalar λ :

$$\| \lambda U \| = | \lambda | \ \| U \| \ , \tag{12.1}$$

$$\| U + V \| \leq \| U \| + \| V \| \ . \tag{12.2}$$

The **trace** of a $n \times n$ matrix M is the sum of its diagonal elements:

$$Tr(M) = M_1^1 + M_2^2 + \cdots + M_n^n \ .$$

Of course, we have:

$$Tr(M^T) = Tr(M) \ ,$$

and if M is skew-symmetric, its trace vanishes. We verify that for a n-row Φ and n-columns U, V :

$$Tr(V\Phi) = \Phi V \qquad \text{and} \qquad Tr(VU^T) = U \cdot V \ . \tag{12.3}$$

and that for any $n \times p$ matrix M and any $p \times n$ matrix N:

$$Tr(MN) = Tr(NM) \ ,$$

The **determinant** of a $n \times n$ matrix M is the unique numerical function det of M, linear with respect to each of its columns $M_1, M_2, \cdots M_n$, completely skew-symmetric with respect to them and such that:

$$\det(1_{\mathbb{R}^n}) = 1 \ .$$

The determinant of a 2×2 matrix is:

$$\det(M) = M_1^1 M_2^2 - M_1^2 M_2^1 \ ,$$

and the one of a 3×3 matrix is:

$$\det(M) = M_1^1 M_2^2 M_3^3 + M_1^2 M_2^3 M_3^1 + M_1^3 M_2^1 M_3^2 - M_1^3 M_2^2 M_3^1 - M_1^2 M_2^1 M_3^3 - M_1^1 M_2^3 M_3^2 \ .$$

If M, N are $n \times n$ matrix, we verify that:

$$\det(-M) = (-1)^n \det(M) \qquad \text{and} \qquad \det(MN) = \det(M)\det(N) \ .$$

A $n \times n$ matrix M is regular if and only if its determinant is not null. Then:

$$\det(M^{-1}) = (\det(M))^{-1} \ .$$

The **dot product** of two $n \times n$ matrices M and N is the scalar:

$$M : N = Tr(M^T N) \ . \tag{12.4}$$

The dot product is commutative. The **norm** of a $n \times n$ matrix is:

$$\| M \| = \sqrt{M : M} \ . \tag{12.5}$$

The matrix norm has similar properties to the ones of the vector norm, in particular 12.1 and 12.2.

The **inverse** of a $n \times n$ matrix M is –if there exists– the unique matrix M^{-1} such that:

$$M\,M^{-1} = M^{-1}M = 1_{\mathbb{R}^n}. \tag{12.6}$$

Then, M is said **regular**. A matrix is regular if and only if its determinant is nonzero.

The (i, j) **minor** of the $n \times n$ matrix M, denoted $minor(i, j, M)$, is the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting the i-th row and j-th column of M. The **adjugate** of M is the $n \times n$ matrix $adj(M)$ of which the element $(adj(M))_j^i$ is the (j, i) **cofactor** of M:

$$(adj(M))_j^i = (-1)^{i+j} minor(j, i, M) \ .$$

If M is regular, its inverse can be obtained thanks to **Cramer's rule**:

$$M^{-1} = \frac{adj(M)}{\det(M)} \ . \tag{12.7}$$

A $n \times n$ matrix M has a real or complex number **eigenvalue** λ if there is a nonzero **eigenvector** $V \in \mathbb{R}^n$ such that

$$M\,V = \lambda\,V \ .$$

The eigenvalues of M are roots of the **characteristic equation**:

$$\det(M - \lambda 1_{\mathbb{R}^n}) = 0 \ .$$

The matrix M is **diagonalizable** if there is a family of n linearly independent eigenvectors $V_1, V_2, \cdots, V_n$ of respective eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then, considering the $n \times n$ matrix $P = (V_1, V_2, \cdots, V_n)$, one has:

$$P^{-1} M\,P = diag(\lambda_1, \lambda_2, \cdots, \lambda_n) \ .$$

We say that P **diagonalizes** M. A symmetric matrix M is diagonalizable, its eigenvalues are real numbers and its eigenvectors are mutually orthogonal. As they are defined to within a scalar factor, they may be choosen as orthonormal.

12.3 Vector calculus in $\mathbb{R}^3$

To any 3-column u is associated a unique 3×3 skew-symmetric matrix:

$$j(u) = \begin{pmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{pmatrix} \ . \tag{12.8}$$

u is sometimes called the **axial vector** of $j(u)$. The map j is regular, linear and satisfies the following identities:

$$j(j(u)v) = vu^T - uv^T \ , \tag{12.9}$$

$$j(u)j(v) = vu^T - (u \cdot v)1_{\mathbb{R}^3} \ , \tag{12.10}$$

from which one we deduce:

$$Tr(j(u)j(v)) = -2u \cdot v \ , \tag{12.11}$$

and:

$$j(u)j(v) - j(v)j(u) = j(j(u)v) \ . \tag{12.12}$$

The **cross product** of two 3-columns u and v is the 3-column $u \times v$ defined by:

$$u \times v = j(u)v \ .$$

Considering the components, one has:

$$\begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \times \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = \begin{pmatrix} u^2v^3 - u^3v^2 \\ u^3v^1 - u^1v^3 \\ u^1v^2 - u^2v^1 \end{pmatrix} .$$

The cross product is anticommutative: $u \times v = -v \times u$ and is not associative. Of course, $u \times u = 0$ and the cross product of two colinear columns is null. The cross product is distributive over the addition. From 12.9 and 12.10, we deduce the expressions of the **vector triple products**:

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u \ , \tag{12.13}$$

$$u \times (v \times w) = (u \cdot w)v - (v \cdot u)w \ . \tag{12.14}$$

Also, condition 12.12 leads to **Jacobi's identity**:

$$u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0 \ . \tag{12.15}$$

The oriented volumes are measured by the **scalar triple product**, symmetric by circular permutation of its arguments:

$$(u \times v) \cdot w = (w \times u) \cdot v = (v \times w) \cdot u \ ,$$

and skew-symmetric with respect to any couple of arguments:

$$(u \times v) \cdot w = -(v \times u) \cdot w = -(w \times v) \cdot u = (u \times w) \cdot v \ .$$

Hence the scalar triple product vanishes if two of its arguments are identical. From 12.10 we deduce the relation:

$$\| u \times v \|^2 + (u \cdot v)^2 = \| u \|^2 \| v \|^2 \ . \tag{12.16}$$

An **orthogonal matrix** R is a 3×3 matrix preserving the dot products of two 3-columns:

$$(R\ u) \cdot (R\ v) = u \cdot v \ . \tag{12.17}$$

We verify that the inverse of R is its transposed matrix:

$$R^T R = R\ R^T = 1_{\mathbb{R}^3} \ , \tag{12.18}$$

and that:

$$(R\ u) \times (R\ v) = R(u \times v) \ . \tag{12.19}$$

Using the map j, this relation reads:

$$j(R^T u) = R^T j(u) R \tag{12.20}$$

A **rotation** is an orthogonal transformation preserving the oriented volumes, then its determinant is equal to 1. An **Euclidean transformation** is an affine transformation of $\mathbb{R}^3$ preserving the dot product of two vectors and the oriented volumes, then composed of a rotation R and a translation $k \in \mathbb{R}^3$:

$$r = Rr' + k \ .$$

A 3×3 symmetric matrix M is diagonalizable with eigenvalues are real numbers and the corresponding matrix $P = (V_1, V_2, V_3)$ is orthogonal. The vectors V_1, V_2, V_3 are mutually orthogonal and of unit norm.

12.4 Linear algebra

12.4.1 Linear space

A **linear space** (or **vector space**) $\mathcal{T}$ is a nonempty set with two operations, the addition and the multiplication by a scalar. Its elements are called **vectors** and denoted with an arrow: $\vec{\mathbf{U}}, \vec{\mathbf{V}}, \cdots$. The addition is associative, commutative and has a zero $\vec{\mathbf{0}}$:

$$\vec{\mathbf{U}} + \vec{\mathbf{0}} = \vec{\mathbf{U}} \ .$$

Each vector has an opposite one:

$$\vec{\mathbf{U}} + (-\vec{\mathbf{U}}) = \vec{\mathbf{0}} \ .$$

The multiplication by a scalar is distributive over the vector addition and:

$$\lambda(\mu\vec{\mathbf{U}}) = (\lambda\mu)\vec{\mathbf{U}} \ ,$$

$$1\vec{\mathbf{U}} = \vec{\mathbf{U}} \ .$$

If the scalars are real numbers, $\mathcal{T}$ is a real linear space. Unless otherwise specified, linear spaces considered in this book are real. The set $\mathbb{R}^n$ of n-columns, the set $(\mathbb{R}^n)^*$ of n-rows and, more generally, the set $\mathbb{M}_{np}$ of the $n \times p$ matrices are exemples of linear spaces.

A **linear subspace** of $\mathcal{T}$ is a subset that is closed under taking linear combinations. The set $\mathbb{M}_{nn}^{symm}$ of the $n \times n$ symmetric matrices and the set $\mathbb{M}_{nn}^{skew}$ of the $n \times n$ skew-symmetric matrices are linear subspaces of $\mathbb{M}_{nn}$.

Let $\mathcal{R}$ be another linear space. A map $\mathbf{A} : \mathcal{T} \to \mathcal{R}$ is a **linear map** if it preserves linear combinations:

$$\mathbf{A}\left(\lambda_1\vec{\mathbf{U}}_1 + \lambda_2\vec{\mathbf{U}}_2 + \cdots + \lambda_p\vec{\mathbf{U}}_p\right) = \lambda_1\mathbf{A}\left(\vec{\mathbf{U}}_1\right) + \lambda_2\mathbf{A}\left(\vec{\mathbf{U}}_2\right) + \cdots + \lambda_p\mathbf{A}\left(\vec{\mathbf{U}}_p\right) \ .$$

A linear space $\mathcal{T}$ has a finite **dimension** n if there exists a linear regular map S of which the definition set is $\mathbb{R}^n$ and the value set is $\mathcal{T}$. S is called a **basis** or **linear**

frame of $\mathcal{T}$. The $\vec{\mathbf{e}}_i = S(e_i)$ are called the **basis vectors**. Thus any vector can be decomposed into the unique linear combination of the basis vectors:

$$\vec{\mathbf{V}} = V^1\vec{\mathbf{e}}_1 + V^2\vec{\mathbf{e}}_2 + \cdots + V^n\vec{\mathbf{e}}_n \ ,$$

where the V^i are called the **components** of $\vec{\mathbf{V}}$ with respect to the basis. We denote it indifferently S or $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \cdots, \vec{\mathbf{e}}_n)$ or more simply $(\vec{\mathbf{e}}_i)$, allowing to write the previous relation:

$$\vec{\mathbf{V}} = S(V) = (\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \cdots, \vec{\mathbf{e}}_n)V \ ,$$

the last expression being understood as a product of a row by a column. If we change the basis $(\vec{\mathbf{e}}_i)$ for a new one $(\vec{\mathbf{e}}'_i)$, the corresponding **transformation matrix** is the regular $n \times n$ matrix $P = S^{-1}S'$ such that:

$$(\vec{\mathbf{e}}'_1, \vec{\mathbf{e}}'_2, \cdots, \vec{\mathbf{e}}'_n) = (\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \cdots, \vec{\mathbf{e}}_n)\, P \ ,$$

that also reads:

$$\vec{\mathbf{e}}'_i = P_i^1\vec{\mathbf{e}}_1 + P_i^2\vec{\mathbf{e}}_2 + \cdots + P_n^n\vec{\mathbf{e}}_n \ .$$

In the new basis, the vector $\vec{\mathbf{V}}$ is represented by the column:

$$V' = P^{-1}V \ . \tag{12.21}$$

Conversely, if there exists a regular map S of which the definition set is $\mathbb{R}^n$, its value set $\mathcal{T}$ is a linear space of dimension n, defining by **structure transport** the vector addition:

$$\vec{\mathbf{U}} + \vec{\mathbf{V}} = S(S^{-1}(\vec{\mathbf{U}}) + S^{-1}(\vec{\mathbf{V}})) \ ,$$

and the multiplication by a scalar:

$$\lambda\vec{\mathbf{U}} = S(\lambda S^{-1}(\vec{\mathbf{U}})) \ .$$

Let $\mathcal{F}$ be the set of the maps f of which the definition set is a given set A and the value set is a given linear space $\mathcal{T}$. Defining for $f, g \in \mathcal{F}$ the vector addition by:

$$\forall x \in A, \qquad (f+g)(x) = f(x) + g(x) \ , \tag{12.22}$$

and the multiplication by a scalar:

$$\forall x \in A, \qquad (\lambda f)(x) = \lambda f(x) \ , \tag{12.23}$$

the set $\mathcal{F}$ is a linear space.

12.4.2 Linear form

The set $\mathcal{T}^*$ of the linear maps from $\mathcal{T}$ into $\mathbb{R}$ is a linear space and is called the **dual space** of $\mathcal{T}$. Its elements $\mathbf{\Phi}$ are called **linear forms** or **covectors**. If $\mathcal{T}$ has a finite dimension n, then its dual one has the same dimension. S or $(\vec{\mathbf{e}}_i)$ being a basis of $\mathcal{T}$, we have:

$$\mathbf{\Phi}(\vec{\mathbf{V}}) = \mathbf{\Phi}(S\,V) = (\mathbf{\Phi}\,S)\,V \ .$$

Hence $\Phi = \mathbf{\Phi}\,S$ is the unique n-row such that $\mathbf{\Phi} = \Phi\,S^{-1}$. The map S^{-1} is called a **cobasis** or a **linear coframe** and the components Φ_i of Φ are the **components** of $\mathbf{\Phi}$ with respect to this cobasis. The $\mathbf{e}^i = e^i S^{-1}$ are called the **basis covectors**. The value of the form $\mathbf{e}^i$ for the vector $\vec{\mathbf{U}}$ is its i-th component in the basis S:

$$\mathbf{e}^i(\vec{\mathbf{U}}) = U^i \ .$$

We denote the cobasis indifferently S^{-1} or $(\mathbf{e}^i)$, allowing to write the previous relation:

$$\mathbf{\Phi} = \Phi\,S^{-1} = \Phi \begin{pmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \vdots \\ \mathbf{e}^n \end{pmatrix} \ ,$$

the last expression being understood as a product of a row by a column. Because:

$$\mathbf{e}^i(\vec{\mathbf{e}}_j) = \delta^i_j \ ,$$

the cobasis S^{-1} is called the **dual basis** or **dual linear frame** of S. We deduce:

$$\mathbf{\Phi}(\vec{\mathbf{e}}_i) = \Phi_i \ . \tag{12.24}$$

In a new cobasis S'^{-1}, the linear form $\mathbf{\Phi}$ is represented by the row:

$$\Phi' = \Phi\,P \ , \tag{12.25}$$

where occurs the transformation matrix $P = S^{-1}S'$. The map:

$$\mathcal{T}^* \times \mathcal{T} \to \mathbb{R} : (\mathbf{\Phi}, \vec{\mathbf{V}}) \mapsto \left\langle \mathbf{\Phi}, \vec{\mathbf{V}} \right\rangle = \mathbf{\Phi}(\vec{\mathbf{V}}) = \Phi\,V \ ,$$

is linear with respect to each of its arguments and is called the **dual pairing**.

12.4.3 Linear map

Let $\mathcal{R}$ be another linear space of finite dimension p and $(\vec{\boldsymbol{\eta}}_i)$ one of its basis. Let $\mathbf{A} : \mathcal{T} \to \mathcal{R} : \vec{\mathbf{U}} \mapsto \vec{\mathbf{V}} = \mathbf{A}\,(\vec{\mathbf{U}})$ be a linear map. If in the basis $(\vec{\boldsymbol{\eta}}_i)$, the vector $\vec{\mathbf{V}}$ is represented by the column V:

$$\vec{\mathbf{V}} = (\vec{\boldsymbol{\eta}}_1, \dots, \vec{\boldsymbol{\eta}}_p)V = \hat{S}(V),$$

and each vector $\mathbf{A}(\vec{\mathbf{e}}_j)$ by the column A_j of components A^i_j, the linear map $\mathbf{A}$ is represented by the matrix:

$A = (A_1, \cdots, A_p)$, $V = AU$.

The linear map $\mathbf{A}$ and the matrix A representing it are linked by:

$$\mathbf{A} = \hat{S}AS^{-1}, \quad A = \hat{S}^{-1}\mathbf{A}\,S.$$

Let Q be the transformation matrix of the change between $(\vec{\boldsymbol{\eta}}_i)$ and a new basis $(\vec{\boldsymbol{\eta}}'_i)$. Then, in the basis $(\vec{\mathbf{e}}_j)$ and $(\vec{\boldsymbol{\eta}}'_i)$, the linear map is represented by the **equivalent matrix**:

$$A' = Q^{-1}AP \ . \tag{12.26}$$

When $\mathcal{T} = \mathcal{R}$, $\mathbf{A}$ is represented by the **similar matrix**:

$$A' = P^{-1}AP \ ,$$

and:

$$Tr(A') = Tr(P^{-1}AP) = Tr(APP^{-1}) = Tr(A) \ ,$$

does not depends of the choice of the basis but only on $\mathbf{A}$. We call it the **trace** of $\mathbf{A}$ and denote it $Tr(\mathbf{A})$. The element A^i_j of the matrix A representing $\mathbf{A}$ in the basis S is given by:

$$A^i_j = \mathbf{e}^i(\mathbf{A}(\vec{\mathbf{e}}_j)) \ ,$$

Hence:

$$Tr(\mathbf{A}) = \sum_{i=1}^{n} \mathbf{e}^i(\mathbf{A}(\vec{\mathbf{e}}_i)) \ . \tag{12.27}$$

The set $Hom(\mathcal{T}, \mathcal{R})$ of the linear maps from $\mathcal{T}$ into $\mathcal{R}$ is a linear space of dimension np. In particular, $Hom(\mathcal{T}, \mathbb{R}) = \mathcal{T}^*$ and $Hom(\mathbb{R}^p, \mathbb{R}^n) = \mathbb{M}_{np}$.

The **transposed** of the linear map $\mathbf{A} : \mathcal{T} \to \mathcal{R}$ is the linear map ${}^t\mathbf{A} : \mathcal{R}^* \to \mathcal{T}^*$ such that:

$$\forall \boldsymbol{\Phi} \in \mathcal{R}^*, \quad \forall \vec{\mathbf{V}} \in \mathcal{T}, \qquad \left\langle \boldsymbol{\Phi}, \mathbf{A}(\vec{\mathbf{V}}) \right\rangle = \left\langle {}^t\mathbf{A}(\boldsymbol{\Phi}), \vec{\mathbf{V}} \right\rangle \ .$$

The transposed map ${}^t\mathbf{A}$ is represented by the transposed matrix A^T of A representing $\mathbf{A}$.

If two maps $\mathbf{A}$ and $\mathbf{B}$ are linear and if:

$$\left\{ \mathbf{B}(\vec{\mathbf{U}}) = \vec{\mathbf{0}} \right\} \qquad \Rightarrow \qquad \left\{ \mathbf{A}(\vec{\mathbf{U}}) = \vec{\mathbf{0}} \right\} \ ,$$

the quotient of $\mathbf{A}$ by $\mathbf{B}$ exists and is linear. The map $\boldsymbol{\lambda} = \mathbf{A}/\mathbf{B}$ is called a **Lagrange's multiplier**, hence:

$$\mathbf{A} = \boldsymbol{\lambda}\,\mathbf{B} \ .$$

12.5 Affine geometry

An **affine space** is a nonempty set $A\mathcal{T}$ of **points**, associated to a linear space $\mathcal{T}$ through a map:

$$A\mathcal{T} \times \mathcal{T} \to A\mathcal{T} : (\mathbf{a}, \vec{\mathbf{U}}) \mapsto \mathbf{a} + \vec{\mathbf{U}} \ ,$$

satisfying the three following conditions:

- for any $\mathbf{a} \in A\mathcal{T}$, $\mathbf{a} + \vec{\mathbf{0}} = \mathbf{a}$,
- for any $\mathbf{a} \in A\mathcal{T}$ and $\vec{\mathbf{U}}, \vec{\mathbf{V}} \in \mathcal{T}$, $(\mathbf{a} + \vec{\mathbf{U}}) + \vec{\mathbf{V}} = \mathbf{a} + (\vec{\mathbf{U}} + \vec{\mathbf{V}})$,
- for any points $\mathbf{a}, \mathbf{b} \in A\mathcal{T}$, there is a unique $\vec{\mathbf{U}} \in \mathcal{T}$ such that $\mathbf{a} + \vec{\mathbf{U}} = \mathbf{b}$.

The unique vector $\vec{\mathbf{U}}$ such that $\mathbf{a} + \vec{\mathbf{U}} = \mathbf{b}$ is denoted by $\overrightarrow{\mathbf{ab}}$. By taking $\mathbf{a}$ as the **origin** in $A\mathcal{T}$, we identify $A\mathcal{T}$ with $\mathcal{T}$ through the regular map $\mathbf{b} \mapsto \overrightarrow{\mathbf{ab}}$.

For any family of m points $(\mathbf{a}_i)$ in $A\mathcal{T}$, for any family of m scalars (λ_i) such that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$, and for any $\mathbf{a} \in A\mathcal{T}$, the point:

$$\mathbf{a} + \lambda_1 \overrightarrow{\mathbf{aa}_1} + \lambda_2 \overrightarrow{\mathbf{aa}_2} + \cdots + \lambda_m \overrightarrow{\mathbf{aa}_m} \ ,$$

signed with the weights λ_i, does not depend on the choice of $\mathbf{a}$ and is called **barycenter** and is denoted

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_m \mathbf{a}_m \ .$$

An **affine subspace** of $A\mathcal{T}$ is a subset that is closed under taking barycenters. For any point $\mathbf{a} \in A\mathcal{T}$ and any subset $W \subset \mathcal{T}$, let $\mathbf{a} + W$ denote the following subset of $A\mathcal{T}$:

$$\mathbf{a} + W = \left\{ \mathbf{a} + \vec{\mathbf{V}} \ \mid \vec{\mathbf{V}} \in W \right\} \ .$$

A nonempty subset A of $A\mathcal{T}$ is an affine subspace if and only if for every point $\mathbf{a} \in A$, the set:

$$W_{\mathbf{a}} = \left\{ \overrightarrow{\mathbf{ab}} \ \mid \mathbf{a} \in A \right\} \ ,$$

is a subspace of $\mathcal{T}$. Consequently, $A = \mathbf{a} + W_{\mathbf{a}}$.

Let $A\mathcal{R}$ be another affine space. A map $\boldsymbol{\alpha} : A\mathcal{T} \to A\mathcal{R}$ is an **affine map** if it preserves barycenters:

$$\boldsymbol{\alpha}\left(\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_p \mathbf{a}_p\right) = \lambda_1 \boldsymbol{\alpha}\left(\mathbf{a}_1\right) + \lambda_2 \boldsymbol{\alpha}\left(\mathbf{a}_2\right) + \cdots + \lambda_p \left(\mathbf{a}_p\right) \ .$$

An affine map preserves affine subspaces and parallelotopes. There exists a unique linear map $\mathbf{A} : \mathcal{T} \to \mathcal{R}$ such that:

$$\boldsymbol{\alpha}\left(\mathbf{a} + \overrightarrow{\mathbf{U}}\right) = \boldsymbol{\alpha}\left(\mathbf{a}\right) + \mathbf{A}\left(\overrightarrow{\mathbf{U}}\right), \qquad \mathbf{A}\left(\overrightarrow{\mathbf{ab}}\right) = \boldsymbol{\alpha}\left(\mathbf{b}\right) - \boldsymbol{\alpha}\left(\mathbf{a}\right) \ .$$

It is called the **linear part** of $\boldsymbol{\alpha}$ and is denoted $\mathbf{A} = lin\left(\boldsymbol{\alpha}\right)$. For instance, the map $a : A\mathbb{R}^n \to A\mathbb{R}^n : V \mapsto V' = C + P\,V$, that may be identified to the couple (C, P), is affine and P is its linear part.

If $\mathcal{T}$ has a finite **dimension** n, we said that $A\mathcal{T}$ has the dimension n. Let $(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n)$ be a set of $(n+1)$ points such that the set of the vectors $\vec{\mathbf{e}}_i = \mathbf{a}_i - \mathbf{a}_0$ is a linear frame. We say that: $(\mathbf{a}_0, (\vec{\mathbf{e}}_i))$ is an affine frame of $A\mathcal{T}$ of origin $\mathbf{a}_0$. For any $\mathbf{a} \in A\mathcal{T}$, the decomposition:

$$\mathbf{a} = \mathbf{a}_0 + V^1\vec{\mathbf{e}}_1 + V^2\vec{\mathbf{e}}_2 + \cdots + V^n\vec{\mathbf{e}}_n \ , \tag{12.28}$$

is unique. We call V^i the (**affine**) **components** of $\mathbf{a}$. In other words, the correspondence between $\mathbf{a} \in A\mathcal{T}$ and the column V collecting the components V^i is one-to-one. This defines a one-to-one affine map : $A\mathbb{R}^n \to A\mathcal{T} : V \mapsto \mathbf{a} = f\,(V)$. We say it is an **affine frame map**. Conversely, let f be a given affine frame map. It defines an affine frame by:

$$\mathbf{a}_0 = f\,(0), \quad \mathbf{a}_i = f\,(e_i), \quad \vec{\mathbf{e}}_i = f\,(e_i) - \mathbf{a}_0 \ ,$$

and (12.28) reads:

$$\mathbf{a} = f\,(V) = \mathbf{a}_0 + S(V) \ , \tag{12.29}$$

where the basis $S = lin\,(f)$ is the linear part of f. If we change the affine frame $(\boldsymbol{a}_{\mathbf{0}}, (\vec{\mathbf{e}}_i))$ for a new one $(\boldsymbol{a}'_0, (\vec{\mathbf{e}}'_i))$, the corresponding transformation matrix being P and $C' = S'^{-1}(\overrightarrow{\boldsymbol{a}'_0\boldsymbol{a}_0})$ being the n-column gathering the components of $\overrightarrow{\boldsymbol{a}'_0\boldsymbol{a}_0}$ in the new basis, the decomposition (12.29) leads to:

$$\mathbf{a} = \mathbf{a}'_0 + \overrightarrow{\boldsymbol{a}'_0\boldsymbol{a}_0} + S(V) = \mathbf{a}'_0 + S'(V')$$

with the transformation law for the components of a point:

$$V' = C' + P^{-1}V \ . \tag{12.30}$$

Compare it to the corresponding relation (12.21) for vectors: points are not vectors!

Conversely, introducing $C = -P\,C'$, one has:

$$V = C + P\,V' \ . \tag{12.31}$$

There is a useful trick to convert this relation into what looks like a linear relation. We add 1 as the $(n+1)$-th component to the vectors V and V', and form the $(n+1) \times (n+1)$ matrix $\tilde{P}$:

$$\tilde{V} = \begin{pmatrix} 1 \\ V \end{pmatrix}, \qquad \tilde{P} = \begin{pmatrix} 1 & 0 \\ C & P \end{pmatrix}, \qquad \tilde{V}' = \begin{pmatrix} 1 \\ V' \end{pmatrix} \ , \tag{12.32}$$

so that (12.31) is equivalent to:

$$\tilde{V} = \tilde{P}\tilde{V}' \ ,$$

The affine maps $\mathbf{\Psi}$ from $A\mathcal{T}$ into $\mathbb{R}$ are called **affine forms** and their set is denoted $A^*\mathcal{T}$. If $A\mathcal{T}$ has a finite dimension n and f is an affine frame map, $\Psi = \mathbf{\Psi} \circ f$ is a affine function from $\mathbb{R}^n$ into $\mathbb{R}$. Hence, it holds:

$$\mathbf{\Psi}(\boldsymbol{a}) = \Psi(V) = \chi + \Phi\, V \ , \tag{12.33}$$

where $\chi = \Psi(0) = \mathbf{\Psi}(\boldsymbol{a}_0)$ and $\Phi = lin(\Psi)$ is a n-row. We call $\Phi_1, \Phi_2, \cdots, \Phi_n, \chi$ the (**affine**) **components** of $\mathbf{\Psi}$. It is convenient to gather the components into a $(n+1)$-row and to identify it to $\tilde{\Psi}$ according to:

$$\Psi(V) = \tilde{\Psi}\, \tilde{V}' = (\chi \quad \Phi) \begin{pmatrix} 1 \\ V \end{pmatrix} .$$

The set $A^*\mathcal{T}$ is a linear space of dimension $(n+1)$ called the **vector dual** of $A\mathcal{T}$. If we change the affine frame $(\boldsymbol{a_0}, (\vec{\mathbf{e}}_i))$ for a new one $(\boldsymbol{a}'_0, (\vec{\mathbf{e}}'_i))$, the components of an affine form are modified according to $\tilde{\Psi}' = \tilde{\Psi}\, \tilde{P}$, that leads to:

$$\chi' = \chi + \Phi\, C, \qquad \Phi' = \Phi\, P \ .$$

It is easy to verify that the inverse transformation law

$$\tilde{\Psi} = \tilde{\Psi}'\, \tilde{P}^{-1} \ , \tag{12.34}$$

reads:

$$\chi = \chi' - \Phi'\, P^{-1} C, \qquad \Phi = \Phi'\, P^{-1} \ .$$

12.6 Vector analysis

12.6.1 Limit and continuity

Let $t \mapsto U(t)$ be a map from an open interval I of $\mathbb{R}$ into $\mathbb{R}^n$. If $t_0 \in I$, we say that $U(t)$ approaches the **limit** $U_0 \in \mathbb{R}^n$ as t approaches t_0 if for any $\varepsilon > 0$ we can find $\eta > 0$ such that

$$\mid t - t_0 \mid < \eta \qquad \Rightarrow \qquad \| \, U(t) - U_0 \, \| < \varepsilon \ ,$$

that reads:

$$\lim_{t \to t_0} U(t) = U_0 \ .$$

Moreover let V be a function valued in $\mathbb{R}^n$ and f be a scalar function in an open interval I, $t_0 \in I$ and:

$$\lim_{t \to t_0} V(t) = V_0, \qquad \lim_{t \to t_0} f(t) = f_0 \ ,$$

then:

$$\lim_{t \to t_0} (U(t) + V(t)) = U_0 + V_0, \qquad \lim_{t \to t_0} (f(t) U(t)) = f_0 U_0 \ .$$

$$\lim_{t \to t_0} (U(t) \cdot V(t)) = U_0 \cdot V_0, \qquad \lim_{t \to t_0} (U(t) \times V(t)) = U_0 \times V_0 \ .$$

The function U is said to be **continuous** at t_0 if:

$$\lim_{t\to t_0} U(t) = U(t_0) \ .$$

It is continuous on the interval $J \subset I$ if it is continuous at every $t \in J$.

The extension of the previous considerations to functions M valued in $\mathbb{M}_{nn}$ is straightforward thanks to the the matrix norm 12.5. If U is a function valued in $\mathbb{R}^n$ and:

$$\lim_{t\to t_0} M(t) = M_0 \ ,$$

we have:

$$\lim_{t\to t_0} (M(t)\, U(t)) = M_0\, U_0 \ ,$$

every time the right hand side exists.

12.6.2 Derivative

We said that U is **differentiable** at t_0 if:

$$\dot{U}(t) = \frac{dU}{dt}(t) = \lim_{t\to t_0} \frac{U(t) - U(t_0)}{t - t_0} \ .$$

exists. This limit is called the **derivative** of U at t_0. We said that U is differentiable on I if it is differentiable at every $t_0 \in I$. Thus, we have:

$$\frac{d}{dt}(fU) = \frac{df}{dt}U + f\frac{dU}{dt}, \qquad \frac{d}{dt}(U \cdot V) = \frac{dU}{dt} \cdot V + u \cdot \frac{dU}{dt} \ ,$$

$$\frac{d}{dt}(U \times V) = \frac{dU}{dt} \times V + U \times \frac{dV}{dt} \ .$$

The extension of the derivative to functions M valued in $\mathbb{M}_{nn}$ is straightforward. Then, we have:

$$\frac{d}{dt}(MU) = \frac{dM}{dt}U + M\frac{dU}{dt} \ .$$

Multiplying by dt, we can adopt the language of differentials:

$$d(fU) = df\, U + f\, dU, \qquad d(U \cdot V) = dU \cdot V + U \cdot dV \ ,$$

$$d(U \times V) = dU \times V + U \times dV, \qquad d(MU) = dM\, U + M\, dU \ .$$

If a scalar function f is differentiable at t_0 and U is differentiable at $f(t_0)$, then the composition $V = Uf$ is differentiable at t_0 with:

$$\frac{d}{dt}(U(f(t))) = \frac{dU}{df}\frac{df}{dt} \ .$$

This is the **chain rule**.

12.7 Partial derivative

Let f be a map from a subset of $\mathbb{R}^n$ into $\mathbb{R}^m$ and v be a n-column. The **directional derivative of f at $x \in def(f)$ in the direction of v** is defined by:

$$Df(x)(v) = \lim_{t \to 0} \frac{f(x + t\,v) - f(x)}{t} \ ,$$

if the limit exists. That is, $Df(x)(v)$ is the ordinary derivative of the function $t \mapsto f(x + t\,v)$ at $t = 0$. If the directional derivative of f at x exists in any direction and $Df(x)$ is linear, we say that f is differentiable at x. The map $Df(x)$ is called the **derivative** of f and, if there is no confusion, we often denote its value $Df(x)\,v$. If f is differentiable at every $x \in def(f)$, we said that f is differentiable. Then it is continuous on $def(f)$. We say that f is **continuously differentiable** (or **of class** C^1) if the map $x \mapsto Df(x)$ is continuous.

Let us consider now **scalar fields** f, *i.e.* such that $val(f) \subset \mathbb{R}$. The **partial derivative of f with respect to x^i at x** is:

$$\frac{\partial f}{\partial x^i}(x) = \partial_i f(x) = Df(x)\,e_i \ .$$

If the fonction $x \mapsto \partial_i f(x)$ has a partial derivative with respect to x^j, we denote it:

$$\frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right) = \frac{\partial^2 f}{\partial x^j \partial x^i} \ .$$

If a function f has continuous partial derivatives up to the order p, we said that it is **of class** C^p. If a function is at least of class C^2, the partial derivatives commute:

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} \ .$$

12.8 Vector analysis

12.8.1 Gradient

The **derivative** of a scalar field at x, denoted:

$$Df(x) = \frac{\partial f}{\partial x} \ ,$$

is a linear map from $\mathbb{R}^n$ into $\mathbb{R}$, that is a n-row and:

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f}{\partial x^1} & \frac{\partial f}{\partial x^2} & \cdots & \frac{\partial f}{\partial x^n} \end{pmatrix} \ .$$

The **gradient** of the scalar field f is the n-column:

$$grad\, f = \left(\frac{\partial f}{\partial x}\right)^T = \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \frac{\partial f}{\partial x^2} \\ \vdots \\ \frac{\partial f}{\partial x^n} \end{pmatrix} .$$

For any scalar fields λ and μ, it holds:

$$grad\,(\lambda\mu) = \lambda\, grad\, \mu + \mu\, grad\, \lambda \ .$$

Let v be a **vector field**, *i.e.* such that $val(v) \subset \mathbb{R}^p$. Its **derivative** at x, denoted:

$$Dv(x) = \frac{\partial v}{\partial x} \ ,$$

is a linear map from $\mathbb{R}^n$ into $\mathbb{R}^p$, that is a $p \times n$ matrix. Its **gradient** is the $n \times p$ matrix:

$$grad\, v = \left(\frac{\partial v}{\partial x}\right)^T \ .$$

For any scalar field λ and any vector fields u, v, one has:

$$grad\,(\lambda v) = grad\, \lambda\, v^T + \lambda\, grad\, v \ . \tag{12.35}$$

$$grad\,(u \cdot v) = (grad\, u)\, v + (grad\, v)\, u \ . \tag{12.36}$$

If $n = p$, the **symmetric gradient** of v is the symmetric $n \times n$ matrix:

$$grad_s\, v = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^T\right] \ .$$

The **skew-symmetric gradient** of v is the skew-symmetric $n \times n$ matrix:

$$grad_a\, v = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \left(\frac{\partial v}{\partial x}\right)^T\right] \ ,$$

If the map:

$$\mathbb{R}^n \to \mathbb{R}^n : x \mapsto y = (grad\, f)\,(x) \ ,$$

is regular, we define **Legendre's transform** of f (with respect to x) as the scalar function:

$$f^*(y) = x \cdot ((grad\, f)\,(x)) - f(x) \ , \tag{12.37}$$

where $x = (grad\, f)^{-1}(y)$. Then the inverse map is:

$$\mathbb{R}^n \to \mathbb{R}^n : y \mapsto x = (grad\, f^*)\,(y) \ .$$

With some abusive notations, Legendre's transform reads:

$$f^* = x \cdot grad\, f - f = \frac{\partial f}{\partial x}\, x - f \ .$$

12.8.2 Divergence

Let v be a **vector field**, such that $val(v), def(v) \subset \mathbb{R}^n$. Its **divergence** is the scalar field:

$$div\, v = Tr\left(\frac{\partial v}{\partial x}\right) \ ,$$

and for any scalar field λ:

$$div\,(\lambda v) = \lambda\, div\, v + \frac{\partial \lambda}{\partial x} v = \lambda\, div\, v + grad\, \lambda \cdot v \ . \tag{12.38}$$

12.8.3 Vector analysis in $\mathbb{R}^3$ and curl

For every vector fields $u, v \in \mathbb{R}^3$, one has:

$$\frac{\partial}{\partial x}(u \times v) = j(u)\, \frac{\partial v}{\partial x} - j(v)\, \frac{\partial u}{\partial x} \ . \tag{12.39}$$

For any column field $r \longmapsto v(r) \in \mathbb{R}^3$ of class C^1, we call **curl** of v the unique 3-column field $curl\, v$ associated to the skew-symmetric gradient of v by the map j:

$$j(curl\, v) = \frac{\partial v}{\partial r} - \left(\frac{\partial v}{\partial r}\right)^T \ . \tag{12.40}$$

For any scalar field λ and any vector field v, one has:

$$curl\,(grad\, \lambda) = 0, \qquad div\,(curl\, v) = 0 \ .$$

Chapter 13

Mathematical tools

13.1 Group

A **group** is a set G together with an **operation** called the **group law**, that we shall denote mutiplicatively:

$$G \times G \to G : (a, b) \mapsto ab \ ,$$

with the following properties:

- **associativity**: $(ab)c = a(bc)$,
- **existence of an identity element** e such that: $\forall a \in G, \qquad ae = ea = a$,
- **existence of an inverse element** a^{-1} for any $a \in G : \ a\,a^{-1} = a^{-1}a = e$.

If the operation is commutative, the group is called **abelian**. For instance, a linear space is an abelian group for the addition with zero as identity element and the opposite vector as inverse element. The set of regular $n \times n$ matrices is a group for the matrix product called the **linear group** and denoted $\mathbb{GL}(n)$. The set of the regular affine transformations of $\mathbb{R}^n$ is called the **affine group** and denoted $\mathbb{A}ff(n)$.

A subset H is a **subgroup** of G if it is also a group for the operation of G. For instance, $\mathbb{GL}(n)$ is a subgroup of $\mathbb{A}ff(n)$. The set of the 3×3 orthogonal matrices is a subgroup of $\mathbb{GL}(3)$ called the **orthogonal group** and denoted $\mathbb{O}(3)$. The set of the rotations of $\mathbb{R}^3$ is a subgroup of $\mathbb{O}(3)$ called the **special orthogonal group** and denoted $\mathbb{SO}(3)$. The Euclidean transformations (resp. special Euclidean transformations) are affine transformation of $\mathbb{R}^3$ of which the linear part is an orthogonal transformation (resp. a rotation). The set of Euclidean transformations (resp. special Euclidean transformations) is a subgroup of $\mathbb{A}ff(3)$ denoted $\mathbb{E}(3)$ (resp. $\mathbb{SE}(3)$).

A **left** (resp. **right**) **action** of G onto a set M is a map:

$$\Phi : G \times M \to M : (a, x) \mapsto x' = \Phi(a, x) = a \bullet x \ ,$$

such that $a \bullet (b \bullet x) = (ab) \bullet x$ (resp. $a \bullet (b \bullet x) = (ba) \bullet x$). A left action in which we substitute a for a^{-1} is a right action. For instance, the map $(P, A) \mapsto A' = P^{-1} A\, P$ is a right action of $\mathbb{GL}(n)$ onto $\mathbb{M}_{nn}$.

Hence the group defines a family of **transformations** (or **symmetries**) of M. We said that G is a **transformation group** (or a **symmetry group**) of M. The **orbital map**:

$$\Phi_x : G \to M : a \mapsto x' = a \bullet x \ ,$$

defines the **variance law**. The **orbit** of x is the value set of the orbital map, *i.e.* the set of all the element of M which can be reached from x through a symmetry:

$$orb(x) = \left\{ x' \quad s.t. \quad \exists a \in G \quad and \quad x' = a \bullet x \right\} \ .$$

A subgroup H of G naturally acts onto a set M by restriction to H of the action of G onto M.

A **linear** (resp. **affine**) **representation** (of finite dimension n) of a group G is a map ρ from G into $\mathbb{GL}(n)$ (resp. $\mathbb{A}ff(n)$) such that:

$$\forall a_1, a_2 \in G, \qquad \rho(a_1 a_2) = \rho(a_1)\, \rho(a_2) \ .$$

13.2 Tensor algebra

13.2.1 Linear tensors

A **tensor** is an object:

- that assigns a set of scalars, called its **components**, to each linear frame S of a linear space $\mathcal{T}$ of finite dimension n,
- with a **transformation law** of these components, when changing of frames, which is a linear representation of $\mathbb{GL}(n)$.

A linear tensor can be constructed as a **multilinear map**, that is a map which is linear with respect to each of its arguments. Let $\mathcal{T}$ and $\mathcal{R}$ be two linear spaces of finite dimensions n and m. We shall be going to define the tensors on $\mathcal{T}$ with values in $\mathcal{R}$. $\mathcal{T}$ is called the **source space** and $\mathcal{R}$ the **target space**. If $\mathcal{R}$ is different of $\mathbb{R}$, the tensor is **vector valued**. If $\mathcal{R} = \mathbb{R}$, we often said more simply that they are tensors on $\mathcal{T}$. The linear tensors can be classified in three families.

The p-**covariant tensors** (or **covariant tensors of rank** p) are the multilinear maps:

$$\mathbf{T} : \overbrace{\mathcal{T} \times \mathcal{T} \times \cdots \mathcal{T}}^{p\ times} \to \mathcal{R} : (\vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2, \cdots, \vec{\mathbf{V}}_p) \mapsto \mathbf{T}(\vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2, \cdots, \vec{\mathbf{V}}_p)$$

As set of maps from the product p times of $\mathcal{T}$ by itself into the linear space $\mathcal{R}$, the set of the p-covariant tensors is a linear space for the operations defined by 12.22

and 12.23. It is denoted $\mathcal{R} \otimes (\otimes^p \mathcal{T}^*)$ or more simple $\otimes^p \mathcal{T}^*$ if $\mathcal{R} = \mathbb{R}$. Hence we generalize the concept of linear forms which are 1-covariants tensors and the one of linear maps from $\mathcal{T}$ into $\mathcal{R}$ which are 1-covariants tensors on $\mathcal{T}$ with values in $\mathcal{R}$.

The q-**contravariant tensors** (or **contravariant tensors of rank** q) are the multilinear maps:

$$\mathbf{T} : \overbrace{\mathcal{T}^* \times \mathcal{T}^* \times \cdots \mathcal{T}^*}^{q\ times} \to \mathcal{R} : (\mathbf{\Phi}_1, \mathbf{\Phi}_2, \cdots, \mathbf{\Phi}_q) \mapsto \mathbf{T}(\mathbf{\Phi}_1, \mathbf{\Phi}_2, \cdots, \mathbf{\Phi}_q)$$

The set of the q-contravariant tensors is a linear space and is denoted $\mathcal{R} \otimes (\otimes^q \mathcal{T})$ or more simple $\otimes^q \mathcal{T}$ if $\mathcal{R} = \mathbb{R}$. The 1-contravariants tensors are linear forms from $\mathcal{T}^*$ into $\mathbb{R}$, then the elements $\hat{\mathbf{V}}$ of the dual space $\mathcal{T}^{**}$ of $\mathcal{T}^*$. It is called the **bidual** space and has the same dimension as $\mathcal{T}^*$ then as $\mathcal{T}$. For any $\vec{\mathbf{V}} \in \mathcal{T}$, the map $\hat{\mathbf{V}}$ defined by $\hat{\mathbf{V}}(\mathbf{\Phi}) = \mathbf{\Phi}(\vec{\mathbf{V}})$ is an element of the bidual verifying:

$$\widehat{\mathbf{V} + \mathbf{U}} = \hat{\mathbf{V}} + \hat{\mathbf{U}}, \qquad \widehat{\lambda \mathbf{V}} = \lambda \hat{\mathbf{V}} \ , \tag{13.1}$$

and the map $\vec{\mathbf{V}} \mapsto \hat{\mathbf{V}}$ is one-to-one from $\mathcal{T}$ into $\mathcal{T}^{**}$. Hence the 1-contravariants tensors can be identified to the vectors. Owing to 12.24, one has:

$$\hat{\mathbf{e}}_i(\mathbf{\Phi}) = \mathbf{\Phi}(\vec{\mathbf{e}}_i) = \Phi_i \ .$$

The **mixed** p-**covariant** and q-**contravariant tensors** are the multilinear maps:

$$\mathbf{T} : \overbrace{\mathcal{T} \times \cdots \mathcal{T}}^{p\ times} \times \overbrace{\mathcal{T}^* \times \cdots \mathcal{T}^*}^{q\ times} \to \mathcal{R} :$$

$$(\vec{\mathbf{V}}_1, \cdots, \vec{\mathbf{V}}_p, \mathbf{\Phi}_1, \cdots, \mathbf{\Phi}_q) \mapsto \mathbf{T}(\vec{\mathbf{V}}_1, \cdots, \vec{\mathbf{V}}_p, \mathbf{\Phi}_1, \cdots, \mathbf{\Phi}_q)$$

The order in which the arguments appear in $\mathbf{T}$ must be specified. To simplify, we choose here to order the arguments starting with all the vectors of $\mathcal{T}$ and following with the ones of $\mathcal{T}^*$. The set of the p-covariant and q-contravariant tensors is a linear space and is denoted $\mathcal{R} \otimes (\otimes^p \mathcal{T}^*) \otimes (\otimes^q \mathcal{T})$ or more simple $(\otimes^p \mathcal{T}^*) \otimes (\otimes^q \mathcal{T})$ if $\mathcal{R} = \mathbb{R}$. Hence we generalize the concept of linear maps from $\mathcal{T}$ into itself which are mixed 1-covariant and 1-contravariants tensors through the identification of the linear map $\mathbf{A}$ with the tensor $\hat{\mathbf{A}}$ defined by $\hat{\mathbf{A}}(\mathbf{\Phi}, \vec{\mathbf{U}}) = \mathbf{\Phi}(\mathbf{A}(\vec{\mathbf{U}}))$.

Let $\mathbf{T}$ and $\mathbf{T}'$ be two arbitrary tensors. For instance, we shall suppose that $\mathbf{T}$ is 1-covariant and 1-contravariant and $\mathbf{T}$ is 1-covariant. It is clear that the scalar $\mathbf{T}(\mathbf{\Phi}, \vec{\mathbf{U}})\, \mathbf{T}'(\vec{\mathbf{V}})$ is linearly depends on $\mathbf{\Phi}, \vec{\mathbf{U}}$ and $\vec{\mathbf{V}}$. Thus the map $\mathbf{T}''$ defined by $\mathbf{T}''(\mathbf{\Phi}, \vec{\mathbf{U}}, \vec{\mathbf{V}})) = \mathbf{T}(\mathbf{\Phi}, \vec{\mathbf{U}})\, \mathbf{T}'(\vec{\mathbf{V}})$ is a 1-covariant and 2-contravariant tensor. $\mathbf{T}''$ is called the **tensor product** of $\mathbf{T}$ and $\mathbf{T}'$ and is denoted $\mathbf{T}'' \otimes \mathbf{T}'$. The generalization of the definition to arbitrary tensors is straightforward. The tensor product is associative but is not in general commutative. It is distributive over the addition.

Let us consider an arbitrary tensor that we shall suppose for instance 3-covariant and 1-contravariant:

$$\mathbf{T}(\vec{\mathbf{U}}, \mathbf{\Phi}, \vec{\mathbf{V}}, \vec{\mathbf{W}})$$

Let us chose a basis S of $\mathcal{T}$. According to the linearity of $\mathbf{T}$, one has:

$$\mathbf{T}(\vec{\mathbf{U}}, \mathbf{\Phi}, \vec{\mathbf{V}}, \vec{\mathbf{W}}) = \mathbf{T}(\sum_i U^i \vec{\mathbf{e}}_i, \sum_j \Phi_j \mathbf{e}^j, \sum_k V^k \vec{\mathbf{e}}_k, \sum_l W^l \vec{\mathbf{e}}_l) \ ,$$

$$\mathbf{T}(\vec{\mathbf{U}}, \mathbf{\Phi}, \vec{\mathbf{V}}, \vec{\mathbf{W}}) = \sum_{ijkl} U^i \Phi_j V^k W^l \mathbf{T}(\vec{\mathbf{e}}_i, \mathbf{e}^j, \vec{\mathbf{e}}_k, \vec{\mathbf{e}}_l) \ .$$

With the convention of summation on the repeated indices which will be omitted in the sequel except explicit mention of the contrary, it holds:

$$\mathbf{T}(\vec{\mathbf{U}}, \mathbf{\Phi}, \vec{\mathbf{V}}, \vec{\mathbf{W}}) = U^i \Phi_j V^k W^l T_i{}^j{}_{kl} \ ,$$

where the n^4 scalars $T_i{}^j{}_{kl} = \mathbf{T}(\vec{\mathbf{e}}_i, \mathbf{e}^j, \vec{\mathbf{e}}_k, \vec{\mathbf{e}}_l)$ are called the **components** of the tensor $\mathbf{T}$ in the basis S. Observing that:

$$U^i \Phi_j V^k W^l = \mathbf{e}^i(\vec{\mathbf{U}})\, \vec{\mathbf{e}}_j(\mathbf{\Phi})\, \mathbf{e}^k(\vec{\mathbf{V}})\, \mathbf{e}^l(\vec{\mathbf{W}}) = (\mathbf{e}^i \otimes \vec{\mathbf{e}}_j \otimes \mathbf{e}^k \otimes \mathbf{e}^l)\, (\vec{\mathbf{U}}, \mathbf{\Phi}, \vec{\mathbf{V}}, \vec{\mathbf{W}}) \ ,$$

we obtain:

$$\mathbf{T} = T_i{}^j{}_{kl}\, \mathbf{e}^i \otimes \vec{\mathbf{e}}_j \otimes \mathbf{e}^k \otimes \mathbf{e}^l \ .$$

The n^4 tensor products $\mathbf{e}^i \otimes \vec{\mathbf{e}}_j \otimes \mathbf{e}^k \otimes \mathbf{e}^l$ form a basis of the linear space of the 3-covariant and 1-contravariant tensors. Let $T'_m{}^r{}_{st}$ be the components of the tensor in another basis $S' = S\,P$. The components of the tensor are modified according to the **transformation law**:

$$T'_m{}^r{}_{st} = P^i_m (P^{-1})^r_j P^k_s P^l_t\, T_i{}^j{}_{kl} \ .$$

It defines a right action of $\mathbb{GL}(n)$ on the set of the **component system** $T = (T_i{}^j{}_{kl})_{1 \leq i,j,k,l \leq n}$ identified to $\mathbb{R}^{n^4}$. Il is a linear representation of $\mathbb{GL}(n)$ of dimension n^4:

$$T' = \rho(P)\, T \ .$$

Fixing the value of the arguments $\vec{\mathbf{U}}$ and $\vec{\mathbf{W}}$ in the previous mixed tensor, the scalar:

$$\lambda = \mathbf{T}(\vec{\mathbf{U}}, \mathbf{\Phi}, \vec{\mathbf{V}}, \vec{\mathbf{W}}) \ ,$$

linearly depends on $\mathbf{\Phi}$ and $\vec{\mathbf{V}}$. Hence, there exists a 1-covariant and 1-contravariant tensor $\check{\mathbf{T}}$ such that $\lambda = \check{\mathbf{T}}(\mathbf{\Phi}, \vec{\mathbf{V}})$, then a linear mapping $\mathbf{A}$ such that $\check{\mathbf{T}} = \hat{\mathbf{A}}$:

$$\forall \mathbf{\Phi} \in \mathcal{T}^*, \vec{\mathbf{V}} \in \mathcal{T}, \qquad \mathbf{T}(\vec{\mathbf{U}}, \mathbf{\Phi}, \vec{\mathbf{V}}, \vec{\mathbf{W}}) = \mathbf{\Phi}(\mathbf{A}(\vec{\mathbf{V}})) \ .$$

Owing to 12.27 and using the convention of summation, the trace of $\mathbf{A}$:

$$Tr(\mathbf{A}) = \mathbf{e}^r(\mathbf{A}(\vec{\mathbf{e}}_r)) = \mathbf{T}(\vec{\mathbf{U}}, \mathbf{e}^r, \vec{\mathbf{e}}_r, \vec{\mathbf{W}}) \ ,$$

does not depends on the choice of the basis and linearly depends on $\vec{\mathbf{U}}$ and $\vec{\mathbf{W}}$, then it is a 2-covariant tensor $\bar{\mathbf{T}}$ defined by:

$$\bar{\mathbf{T}}(\vec{\mathbf{U}}, \vec{\mathbf{W}}) = \mathbf{T}(\vec{\mathbf{U}}, \mathbf{e}^r, \vec{\mathbf{e}}_r, \vec{\mathbf{W}}) \ ,$$

independant of the choice of the basis and called a **contracted tensor** of $\mathbf{T}$. Its components are:

$$\bar{T}_{mt} = T^r_{mrt} \ ,$$

hence the rule: we give the same value r to a superior index and to an inferior one, next we sum for r varying from 1 to n.

Let us consider a linear map $\mathbf{R}$ mapping a 2-covariant tensor

$$\mathbf{T} = T_{kl}\mathbf{e}^k \otimes \mathbf{e}^l \ ,$$

onto a 2-contravariant tensor:

$$\mathbf{R}(\mathbf{T}) = T_{kl}\mathbf{R}(\mathbf{e}^k \otimes \mathbf{e}^l) \ .$$

Thus, putting $\hat{R}^{ijkl} = \mathbf{R}(\mathbf{e}^k \otimes \mathbf{e}^l)(\mathbf{e}^i \otimes \mathbf{e}^j)$, its components are:

$$[\mathbf{R}(\mathbf{T})]^{ij} = \hat{R}^{ijkl}T_{kl} \ .$$

Denoting $\hat{\mathbf{R}}$ the tensor admitting the $\hat{R}^{ijkl}$ as components, one has:

$$[\mathbf{R}(\mathbf{T})]^{ij} = \left[\hat{\mathbf{R}} \otimes \mathbf{T}\right]^{ijkl}_{kl} \ ,$$

hence the tensor $\mathbf{R}(\mathbf{T})$ is obtained by contracting twice the tensor $\hat{\mathbf{R}} \otimes \mathbf{T}$. The operation $(\hat{\mathbf{R}}, \mathbf{T}) \mapsto \mathbf{R}(\mathbf{T})$ is called a **contracted product**. The twice contracted product is often simply denoted by $\hat{\mathbf{R}} : \mathbf{T}$ and, similarly, the once contracted product by $\hat{\mathbf{R}} \cdot \mathbf{T}$. In particular, the contracted product of a 1-covariant tensor $\mathbf{\Phi}$ and a 1-contravariant tensor $\vec{\mathbf{V}}$ is the value of the linear form $\mathbf{\Phi}$ for the vector $\vec{\mathbf{V}}$:

$$\mathbf{\Phi} \cdot \vec{\mathbf{V}} = \mathbf{\Phi}\,(\vec{\mathbf{V}}) \ .$$

Conversely to **free indices** i, j, the summation indices k, l can be renamed and for this reason they are called **dummy indices**.

The representation of the vector valued tensors is similar. For instance, let us consider a 2-contravariant tensor $\mathbf{T}$. Let $(\vec{\boldsymbol{\eta}}_\alpha)$ be a basis of the target space $\mathcal{R}$. Hence, it reads:

$$\mathbf{T} = \mathbf{T}^\alpha \vec{\boldsymbol{\eta}}_\alpha$$

with:

$$\mathbf{T}^\alpha = T^{ij\alpha}\mathbf{e}^i \otimes \mathbf{e}^j \ .$$

Finally, let us say some words about two important types of tensors. Let T be the $n \times n$ matrix of wich the element at the intersection of the i-th line and j-th column is the component T_{ij} of a 2-covariant tensor $\mathbf{T}$. Then the transformation law reads in matrix form:

$$T' = P^T T\, P \ . \tag{13.2}$$

A 2-covariant tensor $\mathbf{T}$ is **symmetric** (resp. **skew-symmetric**) if:

$$\forall \vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2 \in \mathcal{T}, \qquad \mathbf{T}(\vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2) = \mathbf{T}(\vec{\mathbf{V}}_2, \vec{\mathbf{V}}_1) \qquad (\text{resp.} \quad \mathbf{T}(\vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2) = -\mathbf{T}(\vec{\mathbf{V}}_2, \vec{\mathbf{V}}_1)) \ ,$$

hence:

$$T_{ij} = T_{ji} \qquad (\text{resp.} \quad T_{ij} = -T_{ji}) \ .$$

Let T be the $n \times n$ matrix of which the element at the intersection of the i-th line and j-th column is the component T^{ij} of a symmetric 2-contravariant tensor $\mathbf{T}$. Then the transformation law reads in matrix form:

$$T' = P^{-1} T \, P^{-T} \ . \tag{13.3}$$

A 2-contravariant tensor $\mathbf{T}$ is **symmetric** (resp. **skew-symmetric**) if:

$$\forall \mathbf{\Phi}_1, \mathbf{\Phi}_2 \in \mathcal{T}^*, \qquad \mathbf{T}(\mathbf{\Phi}_1, \mathbf{\Phi}_2) = \mathbf{T}(\mathbf{\Phi}_2, \mathbf{\Phi}_1) \qquad (\text{resp.} \quad \mathbf{T}(\mathbf{\Phi}_1, \mathbf{\Phi}_2) = -\mathbf{T}(\mathbf{\Phi}_2, \mathbf{\Phi}_1)) \ ,$$

hence:

$$T^{ij} = T^{ji} \qquad (\text{resp.} \quad T^{ij} = -T^{ji}) \ .$$

With the previous conventions, the contracted product of a 2-contravariant tensor $\mathbf{R}$ and a 2-covariant tensor $\mathbf{T}$ is the dot product (12.4) of the matrices R and T gathering their respective components

$$\mathbf{R} : \mathbf{T} = R^{ij} T_{ij} = R : T \ .$$

The extension of tensor Algebra to vector valued tensors is straightforward. For instance, a 1-covariant tensor $\boldsymbol{T}$ defined on the linear space $\mathcal{T}$ of dimension n with vector values in the linear space $\mathcal{R}$ of dimension p is a linear map from $\mathcal{T}$ into $\mathcal{R}$. Let $(\vec{\mathbf{e}}_j)$ be a basis of $\mathcal{T}$ and $(\boldsymbol{\eta}^i)$ a cobasis of $\mathcal{R}$. if $\vec{\mathbf{V}} = \mathbf{T}\,(\vec{\mathbf{U}})$, then because of the linearity of $\boldsymbol{T}$ and $\boldsymbol{\eta}^i$:

$$V^i = \boldsymbol{\eta}^i(\boldsymbol{T}(U^j \vec{\mathbf{e}}_j)) = T^i_j U^j \ ,$$

where the np components $T^i_j = \boldsymbol{\eta}^i(\boldsymbol{T}(\vec{\mathbf{e}}_j))$ of the tensor are the elements of the matrix T representing the linear map in the considered basis. Let P (resp. Q) be the transformation matrix of the change between $(\vec{\boldsymbol{e}}_j)$ and a new basis $(\vec{\boldsymbol{e}}'_j)$ (resp. $(\vec{\boldsymbol{\eta}}_i)$ and $(\vec{\boldsymbol{\eta}}'_i)$). As the linear map is represented in the new basis by the **equivalent matrix** (12.26):

$$T' = Q^{-1} T P \ , \tag{13.4}$$

the transformation law of the tensor reads in indicial notation:

$$T'^r_s = (Q^{-1})^r_i P^j_s T^i_j \tag{13.5}$$

13.2.2 Affine tensors

An **affine tensor** is an object:

- that assigns a set of **components** to each affine frame f of an affine space $A\mathcal{T}$ of finite dimension n,
- with a **transformation law**, when changing of frames, which is an affine or a linear representation of $\mathbb{A}ff(n)$.

With this definition, the affine tensors are a natural generalization of the classical tensors defined at the previous section and that we shall call **linear tensors**, these last ones being trivial affine tensors for which the affine transformation $a = (C, P)$ acts through its linear part $P = lin(a)$.

An affine tensor can be constructed as a map which is affine or linear with respect to each of its arguments. As the linear tensors, the affine ones can be classified in three families.

The basic p-**covariant affine tensors** (or **covariant affine tensors of rank** p) are the multiaffine maps:

$$\mathbf{T} : \overbrace{A\mathcal{T} \times A\mathcal{T} \times \cdots A\mathcal{T}}^{p\ times} \rightarrow \mathcal{R} : (\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_p) \mapsto \mathbf{T}(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_p)$$

The set of the p-covariant affine tensors is a linear space and is denoted $\mathcal{R} \otimes (\otimes^q A^* \mathcal{T})$ or more simple $\otimes^q A^* \mathcal{T}$ if $\mathcal{R} = \mathbb{R}$. They generalize the affine forms which are 1-covariant affine tensors. Other kinds of affine tensors can be generated by taking linear parts. For instance, from a 2-covariant affine tensor $\mathbf{T}$, we can derive other 2-covariant affine tensors:

- the two linear part, $lin_1(\mathbf{T}) : \mathcal{T} \times A\mathcal{T} \rightarrow \mathbb{R}$ and $lin_2(\mathbf{T}) : A\mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}$ such that:
$$(lin_1(\mathbf{T}))(\overrightarrow{\mathbf{a}_1\mathbf{b}_1}, \mathbf{a}_2) = \mathbf{T}(\mathbf{b}_1, \mathbf{a}_2) - \mathbf{T}(\mathbf{a}_1, \mathbf{a}_2)\ ,$$
$$(lin_2(\mathbf{T}))(\mathbf{a}_1, \overrightarrow{\mathbf{a}_2\mathbf{b}_2}) = \mathbf{T}(\mathbf{a}_1, \mathbf{b}_2) - \mathbf{T}(\mathbf{a}_1, \mathbf{a}_2)\ .$$
- By taking the linear part of $lin_1(\mathbf{T})$ as function of its second argument (or the linear part of $lin_2(\mathbf{T})$ as function of its first argument), we obtain the **bilinear part** of $\mathbf{T}$, a 2-covariant linear tensor such that:
$$(lin_{12}\mathbf{T})(\overrightarrow{\mathbf{a}_1\mathbf{b}_1}, \overrightarrow{\mathbf{a}_2\mathbf{b}_2}) = \mathbf{T}(\mathbf{b}_1, \mathbf{b}_2) - \mathbf{T}(\mathbf{b}_1, \mathbf{a}_2) - \mathbf{T}(\mathbf{a}_1, \mathbf{b}_2) + \mathbf{T}(\mathbf{a}_1, \mathbf{a}_2)\ .$$

The basic q-**contravariant affine tensors** (or **contravariant affine tensors of rank** q) are the multilinear maps:

$$\mathbf{T} : \overbrace{A^*\mathcal{T} \times A^*\mathcal{T} \times \cdots A^*\mathcal{T}}^{q\ times} \rightarrow \mathcal{R} : (\mathbf{\Psi}_1, \mathbf{\Psi}_2, \cdots, \mathbf{\Psi}_q) \mapsto \mathbf{T}(\mathbf{\Psi}_1, \mathbf{\Psi}_2, \cdots, \mathbf{\Psi}_q)$$

The set of the q-contravariant tensors is a linear space and is denoted $\mathcal{R} \otimes (\otimes^q A^{**} \mathcal{T})$ or more simple $\otimes^q A^{**} \mathcal{T}$ if $\mathcal{R} = \mathbb{R}$. A particular attention is paid to the most simple ones, the 1-contravariant affine tensors $\boldsymbol{T}$ with scalar values of which the set is the dual $A^{**}\mathcal{T} = (A^*\mathcal{T})^*$ of the affine dual space, then a linear space of dimension $(n+1)$. As above, we claim that $\boldsymbol{T}(\mathbf{\Psi}) = \mathbf{\Psi}(\boldsymbol{T})$. We denote $\mathbf{1}$ the constant function defined by $\mathbf{1}(\boldsymbol{a}) = 1$. Among the elements of $A^{**}\mathcal{T}$, we can focus our attention on the two following kinds:

- For any $\vec{\boldsymbol{U}} \in \mathcal{T}$, the map $\hat{\boldsymbol{U}}$ defined by $\hat{\boldsymbol{U}}(\boldsymbol{\Psi}) = (lin(\boldsymbol{\Psi}))(\vec{\boldsymbol{U}})$ is such that $\mathbf{1}(\hat{\boldsymbol{U}}) = \hat{\boldsymbol{U}}(\mathbf{1}) = 0$ and verifies (13.1). Hence the vectors of $\mathcal{T}$ can be identified to the elements of the linear subspace of equation $\mathbf{1}(\boldsymbol{T}) = 0$.

- For any $\boldsymbol{a} \in A\mathcal{T}$, the map $\hat{\boldsymbol{a}}$ defined by $\hat{\boldsymbol{a}}(\boldsymbol{\Psi}) = \boldsymbol{\Psi}(\boldsymbol{a})$ is such that $\mathbf{1}(\boldsymbol{a}) = 1$. If $\boldsymbol{b} = \boldsymbol{a} + \vec{\boldsymbol{U}}$, one has:

$$\hat{\boldsymbol{b}}(\boldsymbol{\Psi}) = \boldsymbol{\Psi}(\boldsymbol{a} + \vec{\boldsymbol{U}}) = \boldsymbol{\Psi}(\boldsymbol{a}) + (lin(\boldsymbol{\Psi}))(\vec{\boldsymbol{U}}) = \hat{\boldsymbol{a}}(\boldsymbol{\Psi}) + \hat{\boldsymbol{U}}(\boldsymbol{\Psi}) = (\hat{\boldsymbol{a}} + \hat{\boldsymbol{U}})(\boldsymbol{\Psi})$$

  As the affine form $\boldsymbol{\Psi}$ is arbitrary, this define an action $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{U}}) \mapsto \hat{\boldsymbol{a}} + \hat{\boldsymbol{U}}$. Hence the points of $A\mathcal{T}$ can be identified to the elements of the affine subspace of equation $\mathbf{1}(\boldsymbol{T}) = 1$.

Unlike the bidual $\mathcal{T}^{**}$ which can be identified to $\mathcal{T}$, the vector space $A^{**}\mathcal{T}$ is distinct from the affine space $A\mathcal{T}$ but contains it, reason for which it is called the **vector hull** of $A\mathcal{T}$.

The **mixed** p**-covariant** and q**-contravariant affine tensors** are the p-affine and q-linear maps:

$$\mathbf{T} : \overbrace{A\mathcal{T} \times \cdots A\mathcal{T}}^{p\ times} \times \overbrace{A^*\mathcal{T} \times \cdots A^*\mathcal{T}}^{q\ times} \to \mathcal{R} :$$

$$(\mathbf{a}_1, \cdots, \mathbf{a}_p, \boldsymbol{\Psi}_1, \cdots, \boldsymbol{\Psi}_q) \mapsto \mathbf{T}(\mathbf{a}_1, \cdots, \mathbf{a}_p, \boldsymbol{\Psi}_1, \cdots, \boldsymbol{\Psi}_q)$$

The order in which the arguments appear in $\mathbf{T}$ must be specified. To simplify, we choose here to order the arguments starting with all the points of $A\mathcal{T}$ and following with all the forms of $A^*\mathcal{T}$. The set of the p-covariant and q-contravariant affine tensors is a linear space and is denoted $\mathcal{R} \otimes (\otimes^p A^*\mathcal{T}) \otimes (\otimes^q A^{**}\mathcal{T})$ or more simple $(\otimes^p A^*\mathcal{T}) \otimes (\otimes^q A^{**}\mathcal{T})$ if $\mathcal{R} = \mathbb{R}$. For instance, the linear maps from $A^*\mathcal{T}$ into $\mathcal{T}^*$ are mixed 1-covariant and 1-contravariants affine tensors through the identification of the linear map $\boldsymbol{\mu}$ with the tensor $\hat{\boldsymbol{\mu}}$ defined by:

$$\hat{\boldsymbol{\mu}}(\vec{\mathbf{U}}, \boldsymbol{\Psi}) = (\boldsymbol{\mu}(\boldsymbol{\Psi}))\vec{\mathbf{U}} \ . \tag{13.6}$$

The generalization to the affine tensors of the concepts of tensor product, contracted tensor and product is straightforward. For instance, the tensor product of a point $\boldsymbol{a}$ and a vector $\vec{\boldsymbol{U}}$ is the 2-contravariant affine tensor $\boldsymbol{a} \otimes \vec{\boldsymbol{U}}$ such that $(\boldsymbol{a} \otimes \vec{\boldsymbol{U}})(\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2) = \hat{\boldsymbol{a}}(\boldsymbol{\Psi}_1)\,\hat{\boldsymbol{U}}(\boldsymbol{\Psi}_2)$.

It is worth to remark that putting for a 1-covariant affine tensor:

$$\boldsymbol{\Psi} = \chi \mathbf{1} + \Phi_i \boldsymbol{e}^i \ , \tag{13.7}$$

we recover (12.33) with the convention:

$$\boldsymbol{e}^i(\boldsymbol{a}) = \boldsymbol{e}^i(\overrightarrow{\boldsymbol{a}_0\boldsymbol{a}}) \ . \tag{13.8}$$

Let us consider now a 2-covariant affine tensor $\mathbf{T}$ and an affine frame f of origin $\mathbf{a}_0$ and basis S. Hence:

$$\boldsymbol{T}(\boldsymbol{a}_1, \boldsymbol{a}_2) = \boldsymbol{T}(\boldsymbol{a}_0 + \overrightarrow{\boldsymbol{a}_0\boldsymbol{a}_1}, \boldsymbol{a}_0 + \overrightarrow{\boldsymbol{a}_0\boldsymbol{a}_2}) \ ,$$

$$\boldsymbol{T}(\boldsymbol{a}_1, \boldsymbol{a}_2) = \boldsymbol{T}(\boldsymbol{a}_0, \boldsymbol{a}_0) + (lin_1(\boldsymbol{T}))(\overrightarrow{\boldsymbol{a}_0\boldsymbol{a}_1}, \boldsymbol{a}_0)$$
$$+(lin_2(\boldsymbol{T}))(\boldsymbol{a}_0, \overrightarrow{\boldsymbol{a}_0\boldsymbol{a}_2}) + (lin_{12}(\boldsymbol{T}))(\overrightarrow{\boldsymbol{a}_0\boldsymbol{a}_1}, \overrightarrow{\boldsymbol{a}_0\boldsymbol{a}_2}) \ .$$

Taking into account $\overrightarrow{\boldsymbol{a}_0\boldsymbol{a}_1} = U_1^i \vec{e}_i$, $\overrightarrow{\boldsymbol{a}_0\boldsymbol{a}_1} = U_2^i \vec{e}_i$ and the linearity, one has:

$$\boldsymbol{T}(\boldsymbol{a}_1, \boldsymbol{a}_2) = T_{00} + U_1^i T_{i0} + U_2^j T_{0j} + U_1^i U_2^j T_{ij} \ ,$$

where:

$$T_{00} = \boldsymbol{T}(\boldsymbol{a}_0, \boldsymbol{a}_0), \qquad T_{i0} = (lin_1(\boldsymbol{T}))(\vec{\boldsymbol{e}}_i, \boldsymbol{a}_0) \ ,$$
$$T_{0j} = (lin_2(\boldsymbol{T}))(\boldsymbol{a}_0, \vec{\boldsymbol{e}}_j), \qquad T_{ij} = (lin_{12}(\boldsymbol{T}))(\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_i) \ .$$

are called the **components** of the affine tensor. Observing that $\mathbf{1}(\boldsymbol{a}_1) = \mathbf{1}(\boldsymbol{a}_2) = 1$, $U_1^i = \boldsymbol{e}^i(\boldsymbol{a}_1)$ and $U_2^i = \boldsymbol{e}^i(\boldsymbol{a}_2)$ with the convention (13.8), we obtain:

$$\boldsymbol{T} = T_{00}\, \mathbf{1} \otimes \mathbf{1} + T_{i0}\, \boldsymbol{e}^i \otimes \mathbf{1} + T_{0j}\, \mathbf{1} \otimes \boldsymbol{e}^j + T_{ij}\, \boldsymbol{e}^i \otimes \boldsymbol{e}^j \ .$$

If the points $\mathbf{a}_1$ and $\mathbf{a}_2$ are respectively represented in a given affine frame f by the $(n+1)$-columns:

$$\tilde{V}_1 = \begin{pmatrix} 1 \\ U_1 \end{pmatrix}, \qquad \tilde{V}_2 = \begin{pmatrix} 1 \\ U_2 \end{pmatrix} \ ,$$

gathering the components of $\mathbf{T}$ into a $(n+1) \times (n+1)$ matrix $\tilde{T}$, its value for $\mathbf{a}_1 = f(U_1)$ and $\mathbf{a}_2 = f(U_2)$ is $\tilde{U}_1^T \tilde{T}\, \tilde{U}_2$. Then the transformation law reads in matrix form:

$$\tilde{T}' = \tilde{P}^T \tilde{T}\, \tilde{P} \ .$$

A 2-covariant affine tensor $\mathbf{T}$ is **skew-symmetric** if:

$$\forall \mathbf{a}_1, \mathbf{a}_2 \in A\mathcal{T}, \qquad \mathbf{T}(\mathbf{a}_1, \mathbf{a}_2) = -\mathbf{T}(\mathbf{a}_2, \mathbf{a}_1) \ .$$

In a similar way, we can study a 2-contravariant affine tensor $\mathbf{T}$. Taking into account the decomposition (13.7) and the linearity, its value reads:

$$\boldsymbol{T}(\boldsymbol{\Psi}, \bar{\boldsymbol{\Psi}}) = \boldsymbol{T}(\chi \mathbf{1} + \Phi_i \boldsymbol{e}^i, \bar{\chi}\mathbf{1} + \bar{\Phi}_i \boldsymbol{e}^i)$$

$$\boldsymbol{T}(\boldsymbol{\Psi}, \bar{\boldsymbol{\Psi}}) = \chi\bar{\chi}\, T^{00} + \Phi_i \bar{\chi}\, T^{i0} + \chi \bar{\Phi}_j\, T^{0j} + \Phi_i \bar{\Phi}_j\, T^{ij} \ .$$

where we defined the components of $\mathbf{T}$:

$$T^{00} = \boldsymbol{T}(\mathbf{1}, \mathbf{1}), \qquad T^{i0} = \boldsymbol{T}(\boldsymbol{e}^i, \mathbf{1}), \qquad T^{0j} = \boldsymbol{T}(\mathbf{1}, \boldsymbol{e}^j), \quad T^{ij} = \boldsymbol{T}(\boldsymbol{e}^i, \boldsymbol{e}^j) \ .$$

Observing that $\chi = \boldsymbol{\Psi}(\boldsymbol{a}_0) = \hat{\boldsymbol{a}}_0(\boldsymbol{\Psi})$ and $\Phi_i = \boldsymbol{\Phi}(\vec{\boldsymbol{e}}_i) = (lin(\boldsymbol{\Psi}))(\vec{\boldsymbol{e}}_i) = \hat{\boldsymbol{e}}_i(\boldsymbol{\Psi})$, it holds:

$$\boldsymbol{T} = T^{00}\, \boldsymbol{a}_0 \otimes \boldsymbol{a}_0 + T^{i0}\, \vec{\boldsymbol{e}}_i \otimes \boldsymbol{a}_0 + T^{0j}\, \boldsymbol{a}_0 \otimes \vec{\boldsymbol{e}}_j + T^{ij}\, \vec{\boldsymbol{e}}_i \otimes \vec{\boldsymbol{e}}_j \ .$$

If the affine forms $\mathbf{\Psi}$ and $\bar{\mathbf{\Psi}}$ are respectively represented in a given affine frame f by the $(n+1)$-rows:

$$\tilde{\Psi} = \begin{pmatrix} \chi & \Phi \end{pmatrix}, \qquad \tilde{\bar{\Psi}} = \begin{pmatrix} \bar{\chi} & \bar{\Phi} \end{pmatrix} ,$$

gathering the components of $\mathbf{T}$ into a $(n+1)\times(n+1)$ matrix $\tilde{T}$, its value is $\tilde{\Psi}\tilde{T}\,\tilde{\bar{\Psi}}^T$. Then the transformation law reads in matrix form:

$$\tilde{T}' = \tilde{P}^{-1}\tilde{T}\,\tilde{P}^{-T} . \tag{13.9}$$

A 2-contravariant affine tensor $\mathbf{T}$ is **skew-symmetric** if:

$$\forall \mathbf{\Psi}_1, \mathbf{\Psi}_2 \in A^*\mathcal{T}, \qquad \boldsymbol{T}(\mathbf{\Psi}_1, \mathbf{\Psi}_2) = -\boldsymbol{T}(\mathbf{\Psi}_2, \mathbf{\Psi}_1) .$$

The reader can easily verify that mixed 1-covariant and 1-contravariants affine tensors $\hat{\boldsymbol{\mu}}$ defined by (13.6) can be decomposed with respect to the affine frame f according to:

$$\hat{\boldsymbol{\mu}} = \mu_{0i}\,\boldsymbol{e}^i \otimes \boldsymbol{a}_0 + \mu_i^j\,\boldsymbol{e}^i \otimes \vec{\boldsymbol{e}}_j ,$$

where the affine components of $\hat{\boldsymbol{\mu}}$ are defined by:

$$\mu_{0i} = \hat{\boldsymbol{\mu}}(\vec{\mathbf{e}}_i, \mathbf{1}), \qquad \mu_i^j = \hat{\boldsymbol{\mu}}(\vec{\mathbf{e}}_i, \boldsymbol{e}^j) .$$

13.2.3 G-tensors and Euclidean tensors

A subgroup G of $\mathbb{A}ff(n)$ naturally acts onto the affine tensors by restriction to G of their transformation law. Let F_G be a set of affine frames of which G is a transformation group. The elements of F_G are called G-**frames**. A G-**tensor** is an object:

- that assigns a set of **components** to G-frame f,
- with a **transformation law**, when changing of frames, which is an affine or a linear representation of G.

The corresponding basis of a G-frame is called a G-**basis**.

For instance, the linear tensors are $\mathbb{GL}(n)$-tensors and the **Euclidean tensors** of $\mathbb{R}^3$ are $\mathbb{E}(3)$-tensors. As they are often used, let us give some more details in a slightly more general framework.

The (**covariant**) **metric tensor** $\boldsymbol{G}$ on a linear space $\mathcal{T}$ is a symmetric 2-covariant tensor which is nondegenerate:

$$\forall \vec{\boldsymbol{U}} \in \mathcal{T}, \qquad \boldsymbol{G}(\vec{\boldsymbol{U}}, \vec{\boldsymbol{V}}) = 0 \quad \Leftrightarrow \quad \vec{\boldsymbol{V}} = \vec{\mathbf{0}} .$$

The value of the metric tensor for $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{V}}$ is called their **scalar product** and denoted $\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{V}}$. The symmetric regular matrix G gathering the components $G_{ij} = \vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j$ is called **Gram's matrix**. Thus, it holds:

$$\boldsymbol{G}(\vec{\boldsymbol{U}}, \vec{\boldsymbol{V}}) = \vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{V}} = U^T G\, V ,$$

where U and V are the columns gathering respectively the components of $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{V}}$. The transformation law reads in matrix form:

$$G' = P^T G\, P \ .$$

A linear space equipped with a metric tensor is called an **Euclidian space**. For instance, $\mathbb{R}^n$ and $\mathbb{M}_{nn}$ equipped with the dot product are Euclidian spaces.

To every vector $\vec{\boldsymbol{U}}$ is associated one and only one linear form $\vec{\boldsymbol{V}} \mapsto \boldsymbol{G}(\vec{\boldsymbol{U}}, \vec{\boldsymbol{V}})$ denoted $\boldsymbol{U}$. The covariant components of $\boldsymbol{U}$ depends on the contravariant components of $\vec{\boldsymbol{U}}$ through the operation of **lowering the index**:

$$U_i = G_{ij} U^j \ .$$

The elements G^{ij} of the inverse G^{-1} of Gram's matrix are the components of a 2-contravariant tensor $\boldsymbol{G}^{-1}$ called **contravariant metric tensor**, hence the reverse operation of **raising the index**:

$$U^i = G^{ij} U_j \ .$$

An **orthogonal basis** is a basis for which Gram's matrix is diagonal. The number p (resp. $q = p - n$) of positive (resp. negative) elements of the diagonal does not depends on the choice of the orthogonal basis. The couple (p, q) is called the **signature** of the metric. An **orthonormal basis** is a basis for which the diagonal form Δ of Gram's matrix is composed of p elements equal to $+1$ and q elements equal to -1 and the other vanish. The set $\mathbb{O}(p, q)$ of the transformation matrices P such that:

$$P^T \Delta P = \Delta \ ,$$

is a subgroup of $\mathbb{GL}(n)$. The set of $\mathbb{E}(p, q)$ of the affine transformations $a = (C, P)$ where $P \in \mathbb{O}(p, q)$ is a subgroup of $\mathbb{A}ff(n)$. Considering the $\mathbb{E}(p, q)$-frames f as being the frames of which the linear part $S = lin(f)$ is orthonormal, we define the $\mathbb{E}(p, q)$-tensors.

If the metric is **positive** ($p = n$), $\mathbb{O}(p, q)$ (resp. $\mathbb{E}(p, q)$) is simply denoted $\mathbb{O}(n)$ (resp. $\mathbb{E}(n)$), that allows to recover as particular case the definition of Euclidean tensors. Therefore, in an orthonormal basis, Gram's matrix is the identity of $\mathbb{R}^n$, that can read $G_{ij} = \delta_{ij}$ where the components of the identity matrix are denoted by Kronecker's covariant symbols δ_{ij}, hence U^i and U_i has the same value.

13.3 Vector analysis

13.3.1 Divergence

Let A be a square matrix field such that $def(A) \subset \mathbb{R}^n, val(A) \subset \mathbb{M}_{nn}$. The **divergence** of A is the field $div\, A \in (\mathbb{R}^n)^*$ of n-rows such that for every uniform vector field $k(x) = C^{te} \in \mathbb{R}^n$:

$$(div\, A)\, k = div\, (A\, k) \ . \tag{13.10}$$

Choosing k as the key-columns, we deduce:

$$div\,(A_1, \cdots, A_n) = (div\,A_1, \cdots, div\,A_n)\ . \qquad (13.11)$$

For any scalar field λ, any vector fields $u, v \in \mathbb{R}^n$, and any square matrix field $A \in \mathbb{M}_{nn}$, it holds:

$$div\,(A\,v) = (div\,A)\,v + Tr\left(A\,\frac{\partial v}{\partial x}\right)\ , \qquad (13.12)$$

$$div\,(\lambda\,A) = \lambda\,div\,A + \frac{\partial \lambda}{\partial x}\,A\ , \qquad (13.13)$$

$$div\ (uv^T) = (div\ u)\ v^T + u^T grad\ v \qquad (13.14)$$

$$div\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}(div\,v)\ . \qquad (13.15)$$

For any open domain $\mathcal{V}$ of $\mathbb{R}^n$ with suitable regularity assumptions and any C^1 vector field v and square matrix field A, we have **Green formulae** (or **divergence formulae**):

$$\int_{\mathcal{V}} div\,v\,d\mathcal{V} = \int_{\partial\mathcal{V}} n^T v\,dS\ , \qquad (13.16)$$

$$\int_{\mathcal{V}} div\,A\,d\mathcal{V} = \int_{\partial\mathcal{V}} n^T A\,dS\ , \qquad (13.17)$$

the column n representing the unit normal vector to $\partial\mathcal{V}$, pointing away from $\mathcal{V}$.

The definition (13.10) of the divergence of a square matrix can be easily generalized to matrices of arbitrary dimensions. Let A be a matrix field such that $def(A) \subset \mathbb{R}^n, val(A) \subset \mathbb{M}_{np}$. The **divergence** of A is the field $div\,A \in (\mathbb{R}^p)^*$ of p-rows such that for every uniform vector field $k(x) = C^{te} \in \mathbb{R}^p$:

$$(div\,A)\,k = div\,(A\,k)\ .$$

Choosing k as the key-columns, we deduce:

$$div\,(A_1, \cdots, A_p) = (div\,A_1, \cdots, div\,A_p)\ . \qquad (13.18)$$

For any vector field $v \in \mathbb{R}^p$, and any matrix field $A \in \mathbb{M}_{np}$, it holds:

$$div\,(A\,v) = (div\,A)\,v + Tr\left(A\,\frac{\partial v}{\partial x}\right)\ , \qquad (13.19)$$

13.3.2 Laplacian

The **laplacian** of a scalar field is the scalar field:

$$\triangle\lambda = div\,(grad\,\lambda)\ .$$

Let v be a **vector field**, such that $val(v), def(v) \subset \mathbb{R}^n$. Its **laplacian** is the vector field:

$$\triangle v = (div\,(grad\,v))^T\ . \qquad (13.20)$$

13.3.3 Vector analysis in $\mathbb{R}^3$ and curl

For any matrix field $r \longmapsto A(r) \in \mathbb{M}_{nn}$ of class C^1, we call **curl** of A the matrix field $r \longmapsto curl\, A(r) \in \mathbb{M}_{nn}$ such that for any $dr, \delta r$:

$$(\delta r)^T dA - (dr)^T \delta A = (dr \times \delta r)^T curl\, A \ , \tag{13.21}$$

where dA (resp. δA) is the infinitesimal variation of A resulting from dr (resp. δr). Alternatively, $curl\, A$ is the matrix field such that for any uniform column field $k(r) = C^{te}$, it holds:

$$curl\,(A\, k) = (curl\, A)\, k \ .$$

13.4 Derivative with respect to a matrix

let $f : \mathbb{M}_{np} \to \mathbb{R} : M \mapsto f(M)$ be a scalar valued matrix function of class C^1. Its derivative is a $p \times n$ matrix $\partial f / \partial M$ defined by:

$$df = Tr\left(\frac{\partial f}{\partial M}\, dM\right) = Tr\left(dM\, \frac{\partial f}{\partial M}\right) \ . \tag{13.22}$$

For instance:

$$\frac{\partial}{\partial M}(\det(M)) = adj(M) \ . \tag{13.23}$$

If M is regular, Cramer's rule (12.7) gives:

$$\frac{\partial}{\partial M}(\det(M)) = \det(M)\, M^{-1} \ . \tag{13.24}$$

Also, differentiating (12.6) gives:

$$d(M^{-1}) = -M^{-1} dM\, M^{-1} \ . \tag{13.25}$$

13.5 Tensor analysis

13.5.1 Differential manifold

A manifold is an object which, locally, just looks like an open subset of Euclidian space, but of which global topology can be quite different. Although many manifolds are realized as subset of Euclidian spaces, the general definition worth reviewing. More precisely, a **manifold** $\mathcal{M}$ of dimension n and class C^p is a topological space equipped with a collection of regular maps ϕ called **coordinate charts** of which (figure 13.1):

- the definition sets are connected open subsets of $\mathbb{R}^n$,
- the value sets are open subsets of $\mathcal{M}$ covering it,

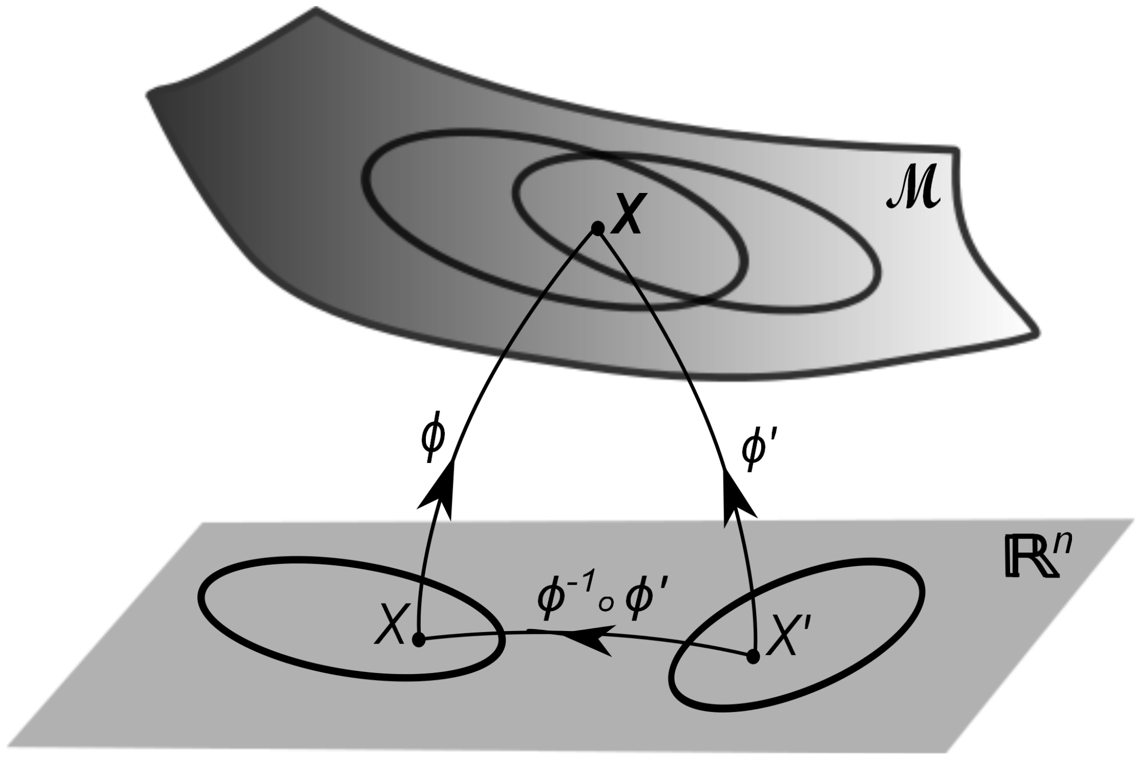

Figure 13.1: coordinate charts of a manifold

- the composite overlap maps $H_{\phi\phi'} = \phi^{-1} \circ \phi'$, called **coordinate changes**, are of class C^p.

They allows to define on $\mathcal{M}$ local coordinate systems:

$$X = \phi^{-1}(\boldsymbol{X}) = \begin{pmatrix} X^1 \\ X^2 \\ \vdots \\ X^n \end{pmatrix} \in \mathbb{R}^n \ .$$

One often expands the collection of coordinate charts to include all possible compatible charts (in the sense that the coordinate changes $X = H_{\phi\phi'}(X')$ are of class C^p), the resulting maximal collection defining an **atlas** on the manifold $\mathcal{M}$. In practice, it is convenient to omit explicit reference to the coordinate map and to identify the points $\boldsymbol{X}$ with their local coordinate expressions X. Nevertheless, objects defined on manifolds must be defined intrinsically, independent of any choice of local coordinates. Consequently, manifolds are suitable tools to develop a coordinate-free approach to the study of their intrinsic geometry.

The basic examples of manifolds are obviously $\mathbb{R}^n$ or any open subset thereof, which are covered by a single chart. Another example is provided by the unit sphere

$S^{n-1} = \{x \in \mathbb{R}^n \quad s.t. \quad \| x \| = 1\}$, which is a manifold of dimension $(n-1)$. It can be covered by two coordinate charts, obtained by omitting the north and south poles respectively. The local coordinates are provided by stereographic projection to $\mathbb{R}^{n-1}$. Alternatively, we can use spherical coordinate on S^{n-1} which are valid away from the poles. A **submanifold** of $\mathcal{M}$ is a subset which, equipped with the topology induced by the one of $\mathcal{M}$, is a manifold in its own right. The simplest examples of submanifolds are curves (of dimension 1) and surfaces (of dimension 2).

A tangent vector to $\mathcal{M}$ at a point $\boldsymbol{X}_0 \in \mathcal{M}$ is geometrically defined by the tangent to a parameterized curve $\lambda \mapsto \boldsymbol{X} = \gamma(\lambda)$ passing through $\boldsymbol{X}_0 = \gamma(0)$. In a coordinate chart ϕ, the curve is represented by $\gamma_\phi = \phi^{-1} \circ \gamma$. We say that two curves γ and $\bar{\gamma}$ are in first order contact at $\boldsymbol{X}_0$ if

$$V = D\gamma_\phi(0) = D\bar{\gamma}_\phi(0) \in \mathbb{R}^n \ .$$

This equivalence relation does not depend on the choice of the coordinate chart and defines an equivalence class denoted $[\gamma]$. As the map $S_\phi(\boldsymbol{X}_0) : V = D\gamma_\phi(0) \mapsto [\gamma]$ is regular, the set of equivalence classes is, by structure transport, a vector space of dimension n called the **tangent space** to $\mathcal{M}$ at $\boldsymbol{X}_0$ and is denoted $T_{\boldsymbol{X}_0}\mathcal{M}$. Its elements are called **tangent vectors** at $\boldsymbol{X}_0$. Hence $S_\phi(\boldsymbol{X}_0)$ is a basis of the tangent space associated to the coordinate chart ϕ. The assigment $\boldsymbol{X} \mapsto S_\phi(\boldsymbol{X})$ defined on the definition set of ϕ is called a **natural frame**. In contrast, an assignment of arbitrary basis which are not related to a coordinate systems is called a **moving frame**. If ϕ' is another coordinate chart, differentiating $\gamma_\phi = H_{\phi\phi'} \circ \gamma_{\phi'}$ with the chain rule provides the transformation law (12.21) of vectors with:

$$P = \frac{\partial H_{\phi\phi'}}{\partial X'}(X'_0) \ ,$$

that is often simply denoted $\partial X / \partial X'$.

The set of linear forms on the tangent space is called the **cotangent space** and is denoted $T^*_{\boldsymbol{X}_0}\mathcal{M}$. The tangent space equipped with a structure of affine space is called the **affine tangent space** and is denoted $AT_{\boldsymbol{X}_0}\mathcal{M}$. Its elements are called **tangent points** at $\boldsymbol{X}_0$. The set of affine forms on the affine tangent space is denoted $A^*T_{\boldsymbol{X}_0}\mathcal{M}$. Of course, if the manifold $\mathcal{M}$ is a vector space (resp. an affine space), it can be identified with the tangent space (resp. affine tangent space) at any of its points.

Let $\mathcal{M}$ an $\mathcal{M}$ be manifolds of respective dimensions n and n'. A map $\boldsymbol{F} : \mathcal{M} \to \mathcal{M}'$ is of class C^p if for any coordinate charts ϕ of $\mathcal{M}$ and ϕ' of $\mathcal{M}'$, the map $F = \phi'^{-1} \circ \boldsymbol{F} \circ \phi$ is of class C^p. The **tangent map** to $\boldsymbol{F}$ at $\boldsymbol{X} \in \mathcal{M}$ is the linear map:

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{X}} : T_{\boldsymbol{X}}\mathcal{M} \to T_{\boldsymbol{F}(\boldsymbol{X})}\mathcal{M}' \ ,$$

represented in the basis S_ϕ and $S_{\phi'}$ by the $n' \times n$ matrix:

$$\frac{\partial F}{\partial X}(X) = S_{\phi'}^{-1} \circ \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{X}} \circ S_\phi \ .$$

Let $\boldsymbol{T}$ be a field of p-covariant tensors on $\mathcal{M}'$. Its **pull-back** by $\boldsymbol{F}$ is the field of p-covariant tensor $\boldsymbol{F}^*\boldsymbol{T}$ on $\mathcal{M}$ defined by:

$$(\boldsymbol{F}^*\boldsymbol{T})(\boldsymbol{X})(\overrightarrow{d_1\mathbf{X}}, \overrightarrow{d_2\mathbf{X}}, \cdots, \overrightarrow{d_p\mathbf{X}}) = \boldsymbol{T}(\boldsymbol{F}(\boldsymbol{X}))\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{X}}\,\overrightarrow{d_1\mathbf{X}}, \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{X}}\,\overrightarrow{d_2\mathbf{X}}, \cdots, \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{X}}\,\overrightarrow{d_p\mathbf{X}}\right) .$$

13.5.2 Covariant differential of linear tensors

Let $\mathcal{M}$ be a manifold of dimension n and class C^2. We call (**linear**) **covariant differential** at $\boldsymbol{X} \in \mathcal{M}$ a map ∇ such that:

- if $\boldsymbol{X} \mapsto \vec{\boldsymbol{T}}(\boldsymbol{X})$ is a tangent vector field and $\overrightarrow{d\boldsymbol{X}}$ is a tangent vector at $\boldsymbol{X}$, $\nabla_{\overrightarrow{d\boldsymbol{X}}}\vec{\boldsymbol{T}}$ is a tangent vector at $\boldsymbol{X}$,
- ϕ being a coordinate chart such that $\boldsymbol{X} = \phi(X)$, there exists a linear map $dX \mapsto \Gamma(dX) \in \mathbb{M}_{nn}$ such that:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}}\vec{\boldsymbol{T}} = S_\phi \nabla_{dX} T \qquad \textit{with} \qquad \nabla_{dX} T = dT + \Gamma(dX)\, T \ . \tag{13.26}$$

The covariant differential being given by Γ' in another coordinate chart ϕ' such that $\boldsymbol{X} = \phi'(X')$, this entails:

$$\Gamma'(dX') = P^{-1}(\Gamma(P\, dX')\, P + dP) \ , \tag{13.27}$$

with $P = \partial X/\partial X'$. By convention, the covariant differential of a scalar field is its usual one. The covariant differential is additive:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}}(\vec{\boldsymbol{T}}_1 + \vec{\boldsymbol{T}}_2) = \nabla_{\overrightarrow{d\boldsymbol{X}}}\vec{\boldsymbol{T}}_1 + \nabla_{\overrightarrow{d\boldsymbol{X}}}\vec{\boldsymbol{T}}_2 \ ,$$

and, $\boldsymbol{X} \mapsto \lambda(\boldsymbol{X})$ being a scalar field:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}}(\lambda\,\vec{\boldsymbol{T}}) = d\lambda\,\vec{\boldsymbol{T}} + \lambda\,\nabla_{\overrightarrow{d\boldsymbol{X}}}\vec{\boldsymbol{T}} \ .$$

When moving from $\boldsymbol{X}$ to the neighbour point $\boldsymbol{X} + d\boldsymbol{X}$, the infinitesimal motion of vectors $\vec{\boldsymbol{e}}_\alpha$ of the basis S_ϕ is given by:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}}\vec{\boldsymbol{e}}_\alpha = S_\phi \nabla_{dX} e_\alpha = S_\phi(\Gamma_\alpha(dX)) = \Gamma^\beta_\alpha \vec{\boldsymbol{e}}_\beta \ . \tag{13.28}$$

As Γ and dT , $\nabla_{dX}T$ is linear with respect to dX. Hence there exists a field $\nabla\vec{\boldsymbol{T}}$ of mixed 1-covariant and 1-contravariant tensors such that:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}}\vec{\boldsymbol{T}} = (\nabla\vec{\boldsymbol{T}}) \cdot \overrightarrow{d\boldsymbol{X}} \ ,$$

and represented in the basis S_ϕ by the matrix:

$$\nabla T = \frac{\partial T}{\partial X} + \Gamma(T) \ . \tag{13.29}$$

As Γ linearly depends on dX, we introduce **Christoffel's symbols** $\Gamma^{\alpha}_{\mu\beta}$ such that:

$$\Gamma^{\alpha}_{\beta}(dX) = \Gamma^{\alpha}_{\mu\beta}\, dX^{\mu} \ . \tag{13.30}$$

A covariant differential is **symmetric** if:

$$\forall dX', \delta X', \qquad \Gamma(dX')\, \delta X' - \Gamma(\delta X')\, dX' = 0 \ ,$$

or, equivalently:

$$\Gamma^{\alpha}_{\mu\beta} = \Gamma^{\alpha}_{\beta\mu} \ . \tag{13.31}$$

In the sequel, the considered covariant differential are assumed symmetric. Putting:

$$\nabla \vec{\boldsymbol{T}} = \nabla_{\beta} T^{\alpha}\, \vec{\boldsymbol{e}}_{\alpha} \otimes \boldsymbol{e}^{\beta} \ ,$$

one has:

$$\nabla_{\beta} T^{\alpha} = \frac{\partial T^{\alpha}}{\partial X^{\beta}} + \Gamma^{\alpha}_{\mu\beta} T^{\mu} \ .$$

The **covariant divergence** of the vector field $\boldsymbol{T}$ is the scalar field obtained by contraction:

$$\boldsymbol{Div}\, \vec{\boldsymbol{T}} = Tr(\nabla \vec{\boldsymbol{T}}) = \nabla_{\alpha} T^{\alpha} = \frac{\partial T^{\alpha}}{\partial X^{\alpha}} + \Gamma^{\alpha}_{\mu\alpha} T^{\mu} \ . \tag{13.32}$$

The extension of the covariant differential to tensors of higher order is straightforward thanks to the rule:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} (\boldsymbol{T} \otimes \boldsymbol{T}') = \nabla_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{T} \otimes \boldsymbol{T}' + \boldsymbol{T} \otimes \nabla_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{T}' \ ,$$

and by contraction:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} (\boldsymbol{T} \cdot \boldsymbol{T}') = (\nabla_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{T}) \cdot \boldsymbol{T}' + \boldsymbol{T} \cdot (\nabla_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{T}') \ . \tag{13.33}$$

For instance, the covariant differential of a 2-contravariant tensor $\boldsymbol{T} = T^{\alpha\beta}\, \vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\beta}$ results from the infinitesimal variation of its components and of the infinitesimal motion of basis vectors when moving from $\boldsymbol{X}$ to the neighbour point $\boldsymbol{X} + d\boldsymbol{X}$:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{T} = dT^{\alpha\beta}\, \vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\beta} + T^{\alpha\beta}\, \nabla_{\overrightarrow{d\boldsymbol{X}}} \vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\beta} + T^{\alpha\beta}\, \vec{\boldsymbol{e}}_{\alpha} \otimes \nabla_{\overrightarrow{d\boldsymbol{X}}} \vec{\boldsymbol{e}}_{\beta} \ ,$$

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{T} = dT^{\alpha\beta}\, \vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}} + T^{\alpha\beta} \Gamma^{\mu}_{\alpha}\, \vec{\boldsymbol{e}}_{\mu} \otimes \vec{\boldsymbol{e}}_{\beta} + T^{\alpha\beta} \Gamma^{\mu}_{\beta}\, \vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\mu} \ ,$$

and, by renaming the dummy indices, we have:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{T} = \nabla_{dX} T^{\alpha\beta}\, \vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\beta} \qquad with \qquad \nabla_{dX} T^{\alpha\beta} = dT^{\alpha\beta} + \Gamma^{\alpha}_{\mu} T^{\mu\beta} + T^{\alpha\mu} \Gamma^{\beta}_{\mu} \ ,$$

or in a matrix form:

$$\nabla_{dX} T = dT + \Gamma(dX) T + T\, (\Gamma(dX))^{T} \ .$$

if $\boldsymbol{X} \mapsto \boldsymbol{\Phi}(\boldsymbol{X})$ is a linear form field and ϕ is a coordinate chart such that $\boldsymbol{X} = \phi(X)$, $\nabla_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{\Phi}$ is a linear form at $\boldsymbol{X}$ such that:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{\Phi} = (\nabla_{dX} \Phi)\, S_\phi^{-1} \qquad \textit{with} \qquad \nabla_{dX} \Phi = d\Phi - \Phi\, \Gamma(dX) \ .$$

The reason of this definition is to be consistent with the rule (13.33) because the contracted product of a linear form and a vector is a scalar field then its covariant differential is the usual one.

$$(\nabla_{dX} \Phi) V + \Phi\, (\nabla_{dX} V) = \nabla_{dX} (\Phi\, V) = d\, (\Phi\, V) \ .$$

When moving from $\boldsymbol{X}$ to the neighbour point $\boldsymbol{X} + d\boldsymbol{X}$, the infinitesimal motion of covectors $\boldsymbol{e}^\alpha$ of the cobasis S_ϕ^{-1} is given by:

$$\nabla_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{e}^\alpha = (\nabla_{dX} e^\alpha)\, S_\phi^{-1} = -(\Gamma^\alpha(dX))\, S_\phi^{-1} = -\Gamma^\alpha_\beta \boldsymbol{e}^\beta \ . \tag{13.34}$$

A **Riemannian metric** on a manifold $\mathcal{M}$ is a field $\boldsymbol{X} \mapsto \boldsymbol{G}(\boldsymbol{X})$ of covariant metric tensor on the tangent spaces. A manifold equipped with a Riemannian metric is called a **Riemannian manifold**. On a Riemannian manifold, there exists one and only one symmetric covariant differential such that the covariant differential of the metric vanishes. Indeed the latter condition reads in any coordinate system:

$$\nabla_{dX} G_{\alpha\beta} = dG_{\alpha\beta} - \Gamma^\mu_\alpha G_{\mu\beta} - G_{\alpha\mu} \Gamma^\mu_\beta = 0 \ ,$$

or, considering the components of these differentials:

$$\frac{\partial G_{\alpha\beta}}{\partial X^\rho} = \Gamma^\mu_{\rho\alpha} G_{\mu\beta} + G_{\alpha\mu} \Gamma^\mu_{\rho\beta} \ .$$

Writing the equations deduced from this latter relation by circular permutation of indices, it holds, owing to (13.31) and the symmetry of the metric tensor:

$$[\rho\alpha, \beta] = \Gamma^\mu_{\rho\alpha} G_{\mu\beta} \ ,$$

where we put:

$$[\rho\alpha, \beta] = \frac{1}{2} \left(\frac{\partial G_{\alpha\beta}}{\partial X^\rho} + \frac{\partial G_{\beta\rho}}{\partial X^\alpha} - \frac{\partial G_{\rho\alpha}}{\partial X^\beta} \right) \ . \tag{13.35}$$

As Gram's matrix is regular, we obtain:

$$\Gamma^\mu_{\alpha\beta} = G^{\mu\rho}\, [\alpha\beta, \rho] \ . \tag{13.36}$$

13.5.3 Covariant differential of affine tensors

The previous tensor analysis can be generalized to the affine tensors. An (**affine**) **covariant differential** at $\boldsymbol{X} \in \mathcal{M}$ is a map $\tilde{\nabla}$ such that:

- if $\boldsymbol{X} \mapsto \boldsymbol{a}(\boldsymbol{X})$ is a tangent point field and $\overrightarrow{d\boldsymbol{X}}$ is a tangent vector at $\boldsymbol{X}$, $\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}} \boldsymbol{a}$ is a tangent vector at $\boldsymbol{X}$,

- V being the components of $\boldsymbol{a}$ in an affine frame f_ϕ of which the basis S_ϕ is its linear part, there exist a linear covariant differential ∇ and a linear map $dX \mapsto \Gamma_A(dX) \in \mathbb{R}^n$ such that:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, \boldsymbol{a} = S_\phi \tilde{\nabla}_{dX} V \qquad \textit{with} \qquad \tilde{\nabla}_{dX} V = \nabla_{dX} V + \Gamma_A(dX) \ .$$

The covariant differential being given by Γ' and Γ'_A in another affine frame $f_{\phi'}$ obtained from f_ϕ through the affine transformation $a = (C, P)$, this entails (13.27) and:

$$\Gamma'_A(dX') = P^{-1}(\Gamma_A(P\, dX') + dC + \Gamma(P\, dX')C) \ . \tag{13.37}$$

Considering only linear transformations $a = (0, P)$ and taking into account the linearity of the map Γ_A, one has:

$$\Gamma'_A = P^{-1} \Gamma_A P \ ,$$

Hence, there exist a mixed 1-covariant and 1-contravariant tensor field $\boldsymbol{A}$ represented in S_ϕ by the matrix $A = \Gamma_A$. Next we can deduce the general expression of Γ_A in any other affine frame obtained through an affine transformation $a = (C, P)$ thanks to its transformation law (13.37) which reads by inversion:

$$\Gamma_A(dX) = P\, \Gamma'_A(dX') - (dC + \Gamma(dX)\, C) = P\, A'\, dX' - (dC + \Gamma(dX)\, C) \ ,$$

or, in short:

$$\Gamma_A(dX) = A\, dX - \nabla_{dX} C \ . \tag{13.38}$$

To be consistent, the symbol $\tilde{\nabla}$ is identified to ∇ when applied to a linear tensor. Moreover, the covariant differential is compatible with the action of the tangent vectors onto the tangent points, according to:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, (\boldsymbol{a} + \vec{\boldsymbol{U}}) = \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, \boldsymbol{a} + \nabla_{\overrightarrow{d\boldsymbol{X}}}\, \vec{\boldsymbol{U}} \ ,$$

or, equivalently:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, \boldsymbol{a}' = \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, \boldsymbol{a} + \nabla_{\overrightarrow{d\boldsymbol{X}}}\, \overrightarrow{\boldsymbol{a}\boldsymbol{a}'} \ .$$

The components of the origin $\mathbf{a}_0$ of the frame f_ϕ vanishing, its infinitesimal motion when moving from $\boldsymbol{X}$ to the neighbour point $\boldsymbol{X} + d\boldsymbol{X}$ is given by:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, \boldsymbol{a}_0 = S_\phi \Gamma_A(dX) = \Gamma^\alpha_A(dX) \vec{\boldsymbol{e}}_\alpha \ . \tag{13.39}$$

Hence, the covariant differential of any tangent point field is:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, \boldsymbol{a} = \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, (\boldsymbol{a}_0 + V^\alpha \vec{\boldsymbol{e}}_\alpha) = \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, \boldsymbol{a}_0 + dV^\alpha \vec{\boldsymbol{e}}_\alpha + V^\alpha \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, \vec{\boldsymbol{e}}_\alpha \ ,$$

that leads to:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\, \boldsymbol{a} = \tilde{\nabla}_{dX}\, V^\alpha \vec{\boldsymbol{e}}_\alpha, \qquad \textit{with} \qquad \tilde{\nabla}_{dX}\, V^\alpha = \nabla_{dX}\, V^\alpha + \Gamma^\alpha_A(dX) \ .$$

There exists a field $\tilde{\nabla}\boldsymbol{a}$ of mixed 1-covariant and 1-contravariant tensors such that:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\,\vec{\boldsymbol{a}} = (\tilde{\nabla}\boldsymbol{a})\cdot\overrightarrow{d\boldsymbol{X}}\ ,$$

As Γ_A linearly depends on dX, we introduce –by analogy with Christoffel's symbols– new symbols $\Gamma^{\alpha}_{A\beta}$ such that:

$$\Gamma^{\alpha}_{A}(dX) = \Gamma^{\alpha}_{A\beta}\,dX^{\beta}\ . \tag{13.40}$$

Thus one obtains:

$$\tilde{\nabla}\boldsymbol{a} = \tilde{\nabla}_{\beta}V^{\alpha}\,\vec{\boldsymbol{e}}_{\alpha}\otimes\boldsymbol{e}^{\beta} \qquad with \qquad \tilde{\nabla}_{\beta}V^{\alpha} = \nabla_{\beta}V^{\alpha} + \Gamma^{\alpha}_{A\beta}\ ,$$

where, according to (13.38):

$$\Gamma^{\alpha}_{A\beta} = A^{\alpha}_{\beta} - \nabla_{\beta}C^{\alpha}\ . \tag{13.41}$$

The **covariant divergence** of the point field $\boldsymbol{a}$ is scalar field:

$$\tilde{\boldsymbol{Div}}\,\boldsymbol{a} = Tr(\tilde{\nabla}\boldsymbol{a}) = \tilde{\nabla}_{\alpha}V^{\alpha} = \nabla_{\alpha}V^{\alpha} + \Gamma^{\alpha}_{A\alpha}\ .$$

The generalization of the covariant differential to affine tensors of higher order results from the rule:

$$\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\,(\boldsymbol{T}\otimes\boldsymbol{T}') = \tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\,\boldsymbol{T}\otimes\boldsymbol{T}' + \boldsymbol{T}\otimes\tilde{\nabla}_{\overrightarrow{d\boldsymbol{X}}}\,\boldsymbol{T}'\ . \tag{13.42}$$

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